6.3 Separation of Variables & Homogeneous Diff Eq.

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Consider: Can we solve?

$dy/dx = (1/2)\,xy$ vs. $dy/dx = (1/2)\,x + y$

$$\frac{dy}{dx} = \frac{1}{2}xy$$

$$dy = \frac{1}{2}xy\,dx$$

$$\frac{dy}{y} = \frac{1}{2}x\,dx$$

$$\frac{dy}{dx} = \frac{1}{2}x + y$$

$$\frac{dy}{y} = \frac{\left(\frac{1}{2}x + y\right)}{y}dx$$

Yes. No.

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14. G: $\sqrt{x} + \sqrt{y}\,\frac{dy}{dx} = 0$

F: solut ∋ $y(1) = 4$

$(1, 4)$

$y^{1/2}\frac{dy}{dx} = -x^{1/2}$

$\frac{dy}{dx} = -\frac{x^{1/2}}{y^{1/2}}$

$dy = -\frac{x^{1/2}}{y^{1/2}}\,dx$

$\int y^{1/2}\,dy = \int -x^{1/2}\,dx$

$\frac{2}{3}y^{3/2} = -\frac{2}{3}x^{3/2} + C$

$2y^{3/2} = -2x^{3/2} + C_1$

$\frac{2}{3}(4)^{3/2} = -\frac{2}{3}\cdot 1 + C$

$\frac{2}{3}\cdot 8 = -\frac{2}{3} + C$

$4^{3/2} = (\sqrt{4})^3$

$\frac{16}{3} + \frac{2}{3} = C$

$C = \frac{18}{3}$

$\frac{3}{3}y^{3/2} = -\frac{2}{3}x^{3/2} + \frac{18}{3}$ $\|$ $2y^{3/2} = -2x^{3/2} + 18$

$y^{3/2} = -x^{3/2} + 9$

6.3 Separation of Variables & Homogeneous Diff Eq.

$$\frac{dy}{dx} = x^3 + 3x^2y - 2y^3$$

Homogeneous Differential Equation

If $M(x,y)\,dx + N(x,y)\,dy = 0$

where M and N are homogeneous functions of the same degree.

then convert to separable solution by substituting $y = vx$

p.423 $f(x,y)$ is homogeneous if

$$f(tx,ty) = t^n f(x,y)$$

28. $f(x,y) = x^3 + 3x^2y^2 - 2y^2$

F: Homogeneous? degree?

$$f(tx,ty) = (tx)^3 + 3(tx)^2(ty)^2 - 2(ty)^2$$

$$t^3x^3 + 3t^2x^2 t^2y^2 - 2t^2y^2$$

$$= t^3x^3 + 3t^4x^2y^2 - 2t^2y^2$$

$$= t^2(\underline{t}x^3 + 3\underline{t^2}x^2y^2 - 2y^2)$$

No Not homogeneous.

30. $\frac{dy}{dx} = f(x,y) = \dfrac{xy}{\sqrt{x^2+y^2}}$

$\sqrt{x^2+y^2}\,dy = xy\,dx$

$\sqrt{x^2+y^2}\,dy - xy\,dx = 0$

$$f(tx,ty) = \frac{tx\cdot ty}{\sqrt{t^2x^2+t^2y^2}} = \frac{t^2xy}{\sqrt{t^2(x^2+y^2)}} = \frac{t^2xy}{t\sqrt{x^2+y^2}} = t\left(\frac{xy}{\sqrt{x^2+y^2}}\right)$$

Yes. Is homogeneous function, but......

No. Not homogeneous diff eq It needs to fit the form:

M(x,y) dx + N(x,y) dy = 0 where M and N are homogeneous to the same degree

30. $f(x,y) = \dfrac{xy}{\sqrt{x^2+y^2}} = \dfrac{dy}{dx}$

$$dy = \frac{xy}{\sqrt{x^2+y^2}}\,dx$$

$$\sqrt{x^2+y^2}\,dy = xy\,dx$$

degree 1

degree 2

Therefore, NOT a homogeneous differential equation.

6.3 Separation of Variables & Homogeneous Diff Eq.

15. G: $y(x+1) + \frac{dy}{dx} = 0$ F: sol. $\ni$ $y(-2) = 1$

$\frac{dy}{dx} = -y(x+1)$

$dy = \frac{dy}{dx}dx = -y(x+1)dx$

$\int \frac{dy}{y} = -\int (x+1)dx$

$\ln|y| = -\left(\frac{x^2}{2} + x\right) + c,$

$e^{-\frac{x^2}{2} - x + c_1} = y = Ce^{-\frac{x^2}{2} - x} = Ce^{-\frac{x^2}{2}}e^{-x} = Ce^{-\left(\frac{x^2}{2} + x\right)}$

$= e^{-\left(\frac{x^2}{2} + x\right)}$

$(-2, 1)$: $1 = Ce^{-(2-2)}$ 1

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9. $\sqrt{1-4x^2}\,\frac{dy}{dx} = x$ F: solve

$$\frac{dy}{dx} = \frac{x}{\sqrt{1-4x^2}}$$

$u = 1-4x^2$

$du = -8x\,dx$

$$\int dy = -\frac{1}{8}\int -8x\,(1-4x^2)^{-1/2}\,dx$$

$$y = -\frac{1}{8}\int u^{-1/2}\,du = -\frac{1}{8}\,\frac{u^{1/2}}{\frac{1}{2}} + C = -\frac{1}{4}\sqrt{1-4x^2} + C$$

6.3 Separation of Variables & Homogeneous Diff Eq.

G: $\frac{dy}{dx} = \frac{x+y}{2x}$

F: solv. ~~log~~ G

$v = \frac{y}{x}$

$dy = \frac{x+y}{2x}dx$

$y = vx$

$dy = v\,dx + x\,dv$

$v\,dx + x\,dv = \frac{x+vx}{2x}dx = \frac{x(1+v)}{2x}dx = \frac{1+v}{2}dx$

$x\,dv = \frac{1+v}{2}dx - v\,dx = \left(\frac{1+v}{2} - v\frac{2}{2}\right)dx = \left(\frac{1+v-2v}{2}\right)dx$

$x\,dv = \frac{1-v}{2}dx$

$-2\int\frac{dv}{1-v} = \int\frac{dx}{x}$

$+2\ln|1-v| = -\ln|x| + c.$

$\ln(1-v)^2 = -\ln|x| + c.$

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$e^{-\ln|x|+c.} = (1-v)^2$

$Ce^{-\ln|x|} = (1-v)^2$

$Ce^{\ln\left|\frac{1}{x}\right|}$

$c\left|\frac{1}{x}\right| = (1-v)^2$

$c\left|\frac{1}{x}\right| = \left(1-\frac{y}{x}\right)^2 = \left(\frac{x-y}{x}\right)^2$

$c|x| = (x-y)^2$

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$$\log_a x = y \Longrightarrow a^y = x$$

$$e^{\ln x} = x$$

$$\ln e^x = x$$

6.3 Separation of Variables & Homogeneous Diff Eq.

39. G: $y' = \frac{xy}{x^2 - y^2}$ F: Solve. $v = \frac{y}{x}$

p.1

let $y = vx$

$dy = v\,dx + x\,dv$

$$\frac{dy}{dx} = \frac{xy}{x^2 - y^2}$$

$$dy = \frac{xy}{x^2 - y^2}\,dx$$

$$v\,dx + x\,dv = \frac{x \cdot vx}{x^2 - (xv)^2}\,dx = \frac{vx^2}{x^2 - v^2x^2}\,dx = \frac{vx^2}{x^2(1-v^2)}\,dx$$

$$v\,dx + x\,dv = \frac{v}{1-v^2}\,dx$$

$$x\,dv = \frac{v}{1-v^2}\,dx - v\,dx = \left(\frac{v}{1-v^2} - v\right)dx$$

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Homework Part 2

6.3 Separation of Variables & Homogeneous Diff Eq.

39. G: $y' = \dfrac{xy}{x^2 - y^2}$ F: Solve. $v = \dfrac{y}{x}$

p.2

let $y = vx$

$dy = v\,dx + x\,dv$

previous:

$$\boxed{x\,dv = \frac{v}{1-v^2}dx - v\,dx = \left(\frac{v}{1-v^2} - v\right)dx}$$

$$= \left[\frac{v}{1-v^2} - v \cdot \frac{1-v^2}{1-v^2}\right]dx$$

$$\frac{1}{x} \cdot \frac{1-v^2}{v^3} \cdot x\,dv = \frac{v - v + v^3}{1-v^2}dx = \frac{v^3}{1-v^2}dx \cdot \frac{1}{x} \cdot \frac{1-v^2}{v^3}$$

$$\frac{1-v^2}{v^3}dv = \frac{dx}{x} \qquad = \left(\frac{1}{v^3} - \frac{v^2}{v^3}\right)dv$$

$$\int \left(v^{-3} - \frac{1}{v}\right)dv = \int \frac{dx}{x}$$

6.3 Separation of Variables & Homogeneous Diff Eq.

39. G: $y' = \dfrac{xy}{x^2 - y^2}$ F: Solve. $v = \dfrac{y}{x}$

p.3

let $y = vx$

$dy = v\,dx + x\,dv$

previous:

$$\int (v^{-3} - \frac{1}{v})\,dv = \int \frac{dx}{x}$$

$$\frac{v^{-2}}{-2} - \ln|v| = \ln|x| + c_1$$

$$-\frac{1}{2v^2} + c_2 = \ln|x| + \ln|v| = \ln|xv|$$

$y = vx$

$v = \frac{y}{x}$

$$-\frac{1x^2}{2y^2} + c_2 = \ln\left|x \cdot \frac{y}{x}\right| = \ln|y|$$

$$\ln|y| = \frac{-x^2}{2y^2} + c_2$$

$$e^{\frac{-x^2}{2y^2} + c_2} = \boxed{y = Ce^{\frac{-x^2}{2y^2}}}$$

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