4.3 Reimann Sums and the Definite Integral

Study 4.3 #13, 17, 19, 23-37, 41, 43

area and definite integral
4.3 Reimann Sums and the Definite Integral

Using the sliders, find the area under the curve, \( y = -x^2 + 4x \), from \( a=0 \) to \( b=4 \). Then find the definite integral from \( a=4 \) to \( b=5 \).

How does the sum of these two values compare to the integral of the function from \( a=0 \) to \( a=5 \)?

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4.3 Riemann Sums and the Definite Integral

Definition of Riemann Sum

Let $f$ be defined on $[a,b]$, and let

$\Delta$ be a partition of $[a,b]$, given by

$a = x_0 < x_1 < x_2 < ... < x_{n-1} < x_n = b$

where $\Delta x_i$ is the width of the $i$th sub-interval

If $c_i$ is any point on the $i$th sub-interval, then

the sum

$$\sum_{i=1}^{n} f(c_i) \Delta x_i, \quad x_{i-1} < c_i < x_i$$

is called a Riemann Sum of $f$ for the partition $\Delta$

$\Delta x_i$ not all equal

norm of the partition $||\Delta|| = \max |\Delta x_i|$ width of longest subinterval
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Definition of Definite Integral

If $f$ is defined on $[a,b]$, and the following limit exists

$$\lim_{||\Delta|| \to 0} \sum_{i=1}^{n} f(c_i) \Delta x_i$$

Then $f$ is integrable on $[a,b]$ and the limit is denoted as:

$$\lim_{||\Delta|| \to 0} \sum_{i=1}^{n} f(c_i) \Delta x_i = \int_{a}^{b} f(x) \, dx$$

This is called the Definite Integral of $f$ from $a$ to $b$
4.3 Reimann Sums and the Definite Integral

\[
\int_0^3 x^3 \, dx = \frac{x^4}{4} \\
\left[ \frac{x^4}{4} \right]_0^4 = \frac{4^4}{4} - \frac{0^4}{4} = 256 \text{ or } 64
\]
4.3 Reimann Sums and the Definite Integral

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Calculus Home Page

Homework Part 1
4.3 Reimann Sums and the Definite Integral

\[ f(x) = 4 - 2x \]

\[ \int_{a}^{b} (4-2x) \, dx \]
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\[ \int_{0}^{4} \frac{x}{2} \, dx \]

Sketch region

Use geometry to find area.

\[ A = \frac{1}{2}bh = \frac{1}{2}(4)(2) = 4 \text{ sq. units} \]
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\[ \int_{0}^{4} \frac{x}{2} \, dx \]

Sketch the region and use geometry to find the area.

\[ A = \frac{1}{2} \cdot 6 \cdot 4 = 12 \text{ sq. units} \]

\[ \int_{0}^{4} \frac{x}{2} \, dx = \frac{1}{2} \left[ \frac{x^2}{2} \right]_{0}^{4} = \frac{1}{4} \left[ 4^2 - 0^2 \right] = 4 \]
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Set up the integral that represents the area

\[ A = \int_{-1}^{1} \frac{1}{x^2 + 1} \, dx \]
4.3 Reimann Sums and the Definite Integral
4.3 Reimann Sums and the Definite Integral

28. 6: \( \int_0^8 (8-x) \, dx \)

\[ A = \frac{1}{2} b h \]
\[ = \frac{1}{2} \cdot 8 \cdot 8 = 32 \]
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Definite Integral as the Area of a Region

If $f$ is continuous and non-negative on $[a,b]$, then the area of the region bounded by:
- the graph of $f$,
- the $x$-axis, and
- the vertical lines $x = a$ and $x = b$

is given by:

$$\text{Area} = \int_a^b f(x) \, dx$$
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Special Integrals

1. \[ \int_a^b f(x) \, dx = 0 \]
2. \[ \int_a^b f(x) \, dx = -\int_b^a f(x) \, dx \]
3. \[ \int_a^b f(x) \, dx = \int_a^c f(x) \, dx + \int_c^b f(x) \, dx \]
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Continuity Implies Integrability

If a function $f$ is continuous on $[a, b]$, then $f$ is integrable on $[a, b]$. 

Continuity & Integrability
4.3 Reimann Sums and the Definite Integral

34. \( \int_{2}^{3} x^3 \, dx = 0 \)

36. \( \int_{2}^{4} 15 \, dx = 15 \int_{2}^{4} \, dx = 15(2) = 30 \)

38. \( \int_{2}^{4} (x^3 + y) \, dx \)

\[ = \int_{2}^{4} x^3 \, dx + \int_{2}^{4} y \, dx = 60 + 8 = 68 \]
4.3 Reimann Sums and the Definite Integral

\[ \int_{a}^{b} f(x) \, dx = \int_{a}^{c} f(x) \, dx + \int_{c}^{b} f(x) \, dx \]

\[ \int_{0}^{3} f(x) \, dx = 4 + 1 = 5 \]

\[ \int_{6}^{3} f(x) \, dx = - (\int_{3}^{6} f(x) \, dx) = -(-1) = 1 \]

\[ \int_{3}^{3} f(x) \, dx = 0 \quad \text{and} \quad \int_{3}^{6} f(x) \, dx = 5(-1) = -5 \]