3.4 Concavity & the 2nd Derivate Test

Study 3.4 #1,5,9,...21; 3,27,31, 35, 39, 51,53,63, 75-78

Goals:
1. Understand how the sign of the 2nd derivative of a function relates to the behavior of the function, re: concave up or concave down.
2. Determine intervals where a function is concave up or concave down.
3. Find Inflection Points of a curve.
4. Use the Second Derivative Test to determine relative extrema.

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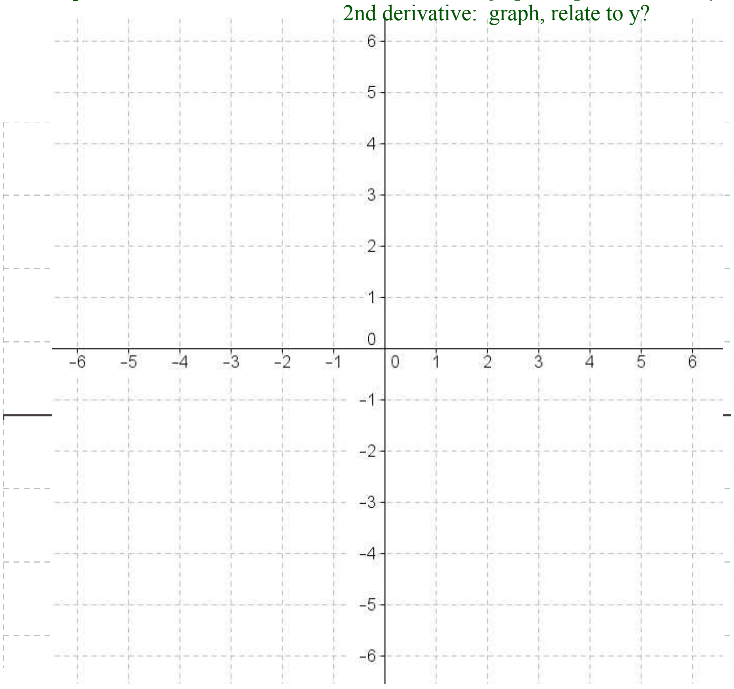
Problems for 3.4

3.4 Concavity & the 2nd Derivate Test

Compare f(x), f '(x), f "(x)

Consider: $y = x^3 - x$

Graph of function
1st Derivative: graph, slope of, relate to y?
2nd derivative: graph, relate to y?



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Problems for 3.4

3.4 Concavity & the 2nd Derivate Test

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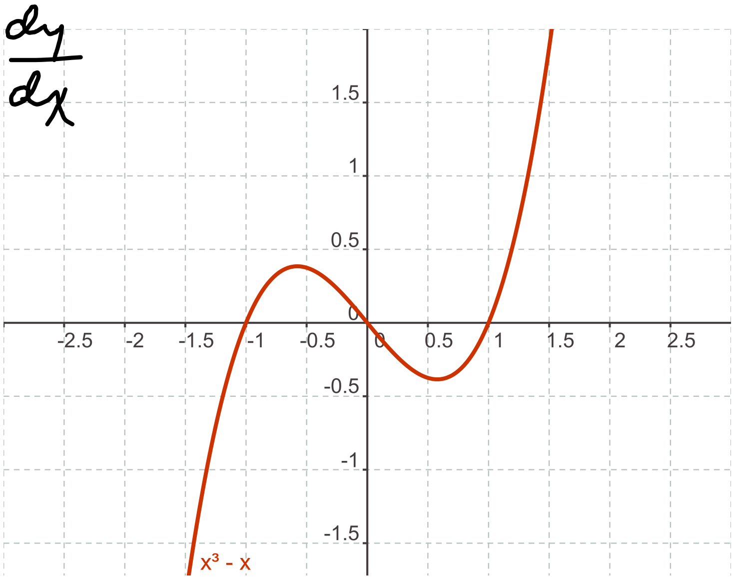
$\frac{dy}{dx} = 3x^2 - 1$

graph $\frac{dy}{dx}$



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Problems for 3.4

3.4 Concavity & the 2nd Derivate Test

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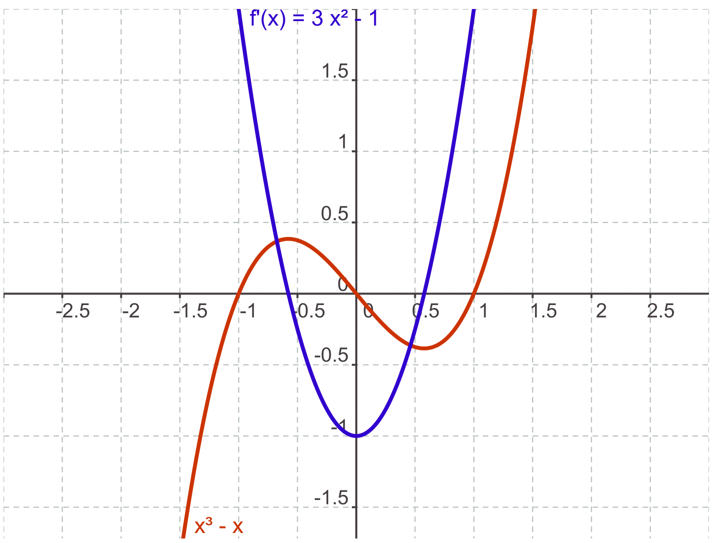

Compare f(x), f '(x), f "(x)

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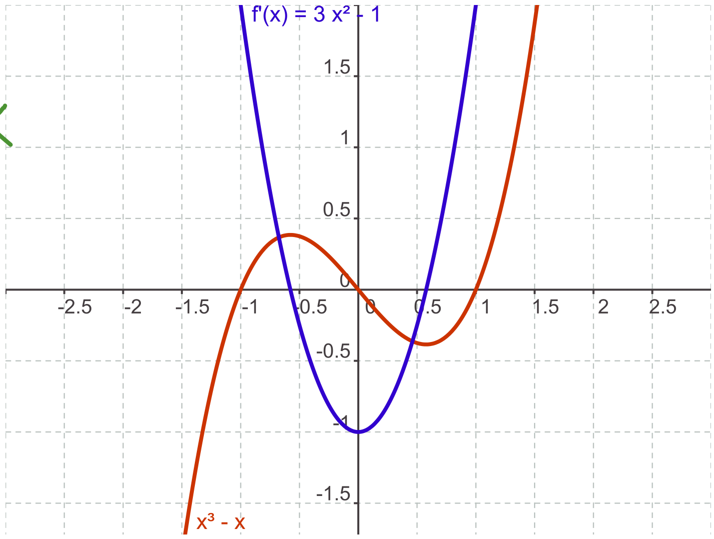
Problems for 3.4

3.4 Concavity & the 2nd Derivate Test

Graph of function
1st Derivative: graph, slope of, relate to y?
2nd derivative: graph, relate to y?

$$\frac{dy}{dx} = 3x^2 - 1$$

$$\frac{d^2y}{dx^2} = 6x$$



Compare f(x), f '(x), f ''(x)

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Problems for 3.4

3.4 Concavity & the 2nd Derivate Test

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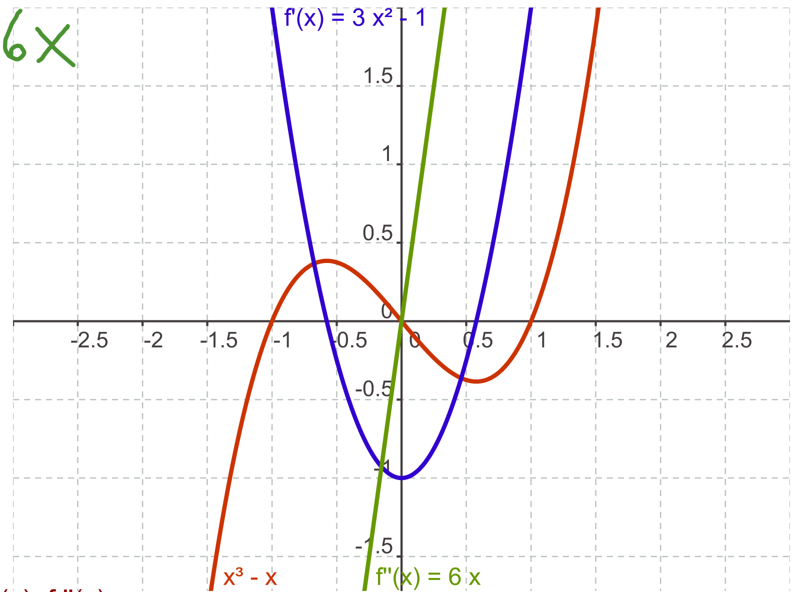

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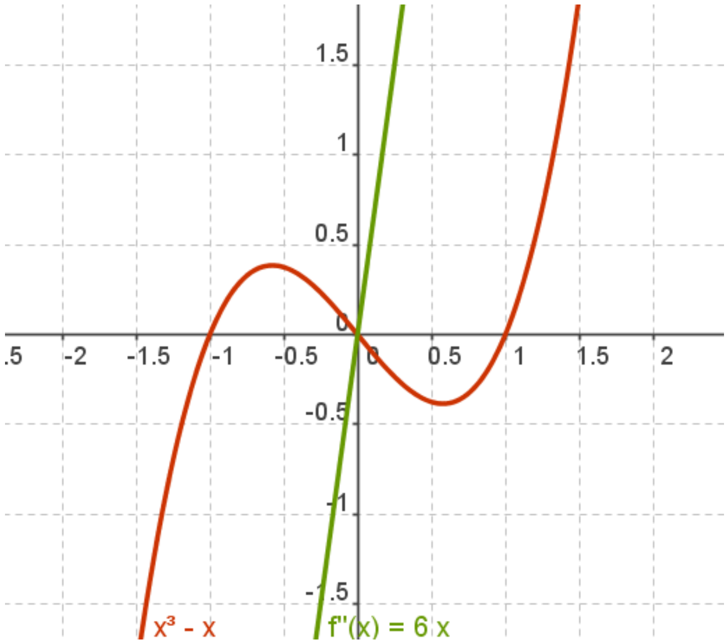
Problems for 3.4

3.4 Concavity & the 2nd Derivate Test

How does the 2nd derivative relate to original function?



C D
concave down
C U
concave up

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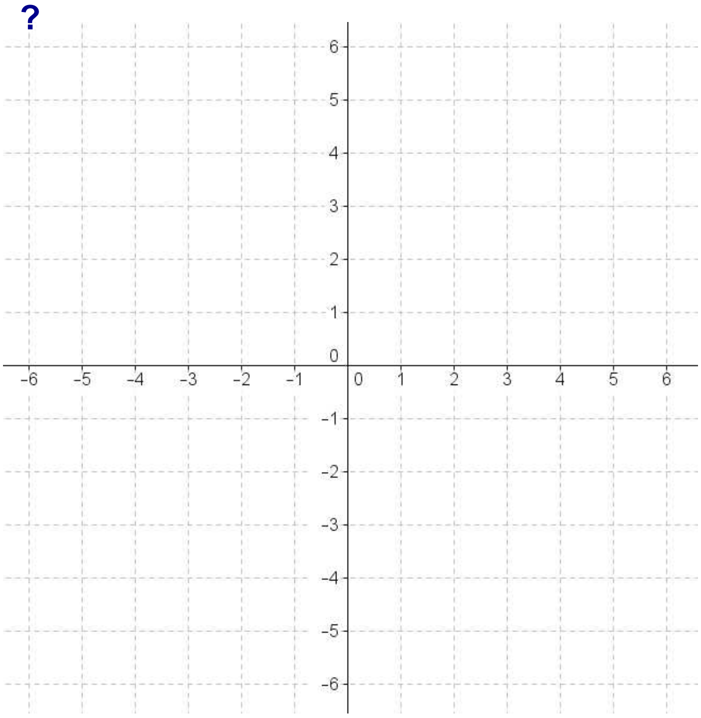
Problems for 3.4

3.4 Concavity & the 2nd Derivate Test

What about $y = x^3$?

$\frac{dy}{dx} =$

$\frac{d^2y}{dx^2} =$



Change f(x) to x^3

C D
concave down
C U
concave up

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Problems for 3.4

3.4 Concavity & the 2nd Derivate Test

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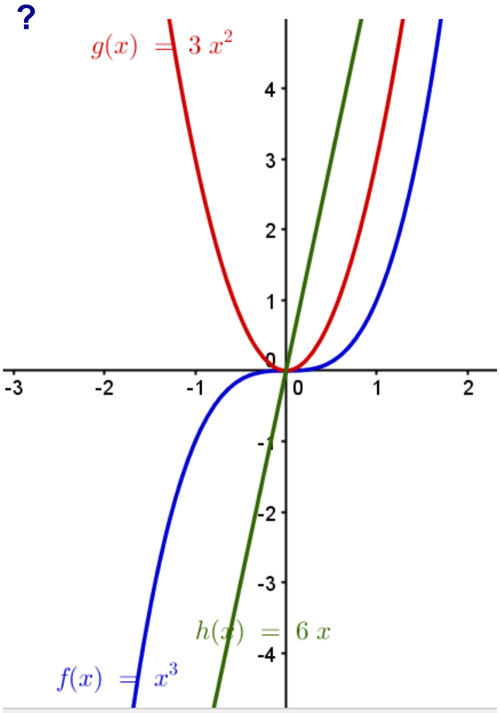

Compare f(x), f '(x), f "(x)

C D concave down C U concave up

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Problems for 3.4

3.4 Concavity & the 2nd Derivate Test

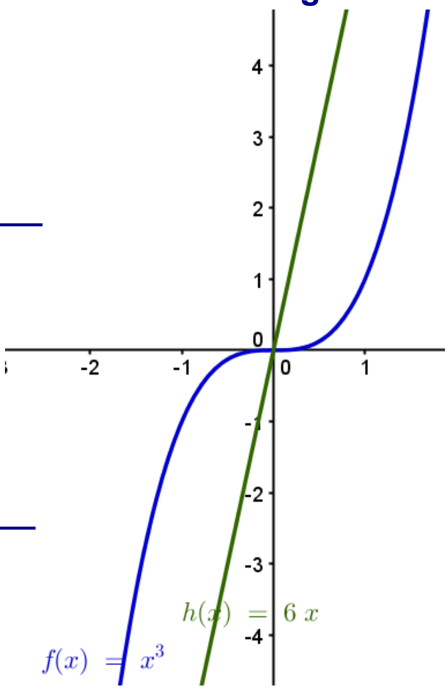
How does the 2nd derivative relate to original function?

When $\frac{d^2y}{dx^2} < 0$?

then y is ___________________

When $\frac{d^2y}{dx^2} > 0$?

then y is ___________________



C D concave down C U concave up

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Problems for 3.4

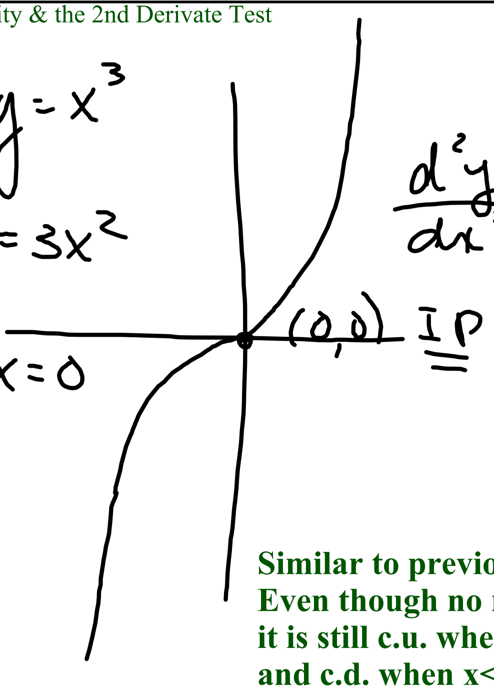
3.4 Concavity & the 2nd Derivate Test

$y = x^3$

$\frac{dy}{dx} = 3x^2$

$\frac{d^2y}{dx^2} = 6x$

CN: $x = 0$



Similar to previous example.
Even though no relative extrema,
it is still c.u. when x>0
and c.d. when x<0.

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Problems for 3.4

3.4 Concavity & the 2nd Derivate Test

Definition of Concavity

Let f be differentiable on an open interval . The graph of *f* is:

concave upward on I if f ' is increasing on I

concave downward on I if f ' is decreasing on I

or when f '' (x) > 0

or when f '' (x) < 0

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Problems for 3.4

2nd Derivative Test for Relative Extrema

Step by step: on-line

Let f be funct. ∋ $f'(c)=0$ (horizontal slope)
and f'' exists on open interval cont. c.

for Rel. Extrema

1. If $f''(c)>0$ (C.U.), then f has rel. min. at $(c, f(c))$

2. If $f''(c)<0$ (C.D.), then f has rel. max at $(c, f(c))$

$f''(c)=0$, the test fails. *

Step by step: on-line

$y = x^3 - x$ F: Rel. extr, 2nd deriv. test

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3.4 Concavity & the 2nd Derivate Test

G: $y = x^3 - x$ F: Rel. extr. 2^{nd} deriv. test

$y = x^3 - x$ $\frac{dy}{dx} = 3x^2 - 1$

$\frac{d^2y}{dx^2} = 3(2x)$

$= 6x$

CNs: $3x^2 = 1$

$x^2 = 1/3$

$x = \pm\sqrt{1/3}$

at $x = \frac{1}{\sqrt{3}}$, $\frac{d^2y}{dx^2} > 0 \therefore$ c.u.

$\left(\frac{1}{\sqrt{3}}, \_\_\right)$ rel. min.

$x = -\frac{1}{\sqrt{3}}$, $\frac{d^2y}{dx^2} < 0 \therefore$ c.d.

$\left(-\frac{1}{\sqrt{3}}, \_\_\right)$ rel. max

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Problems for 3.4

3.4 Concavity & the 2nd Derivate Test

Example: G: $h(x) = x^5 - 5x + 2$

F: open interval where c.u. and c.d.

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Problems for 3.4

3.4 Concavity & the 2nd Derivate Test $-1 + 5 + 2$

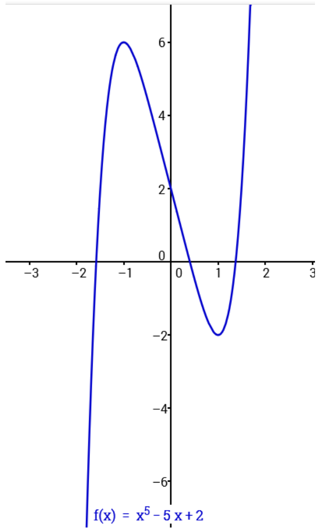
Example: G: $h(x) = x^5 - 5x + 2$
F: open interval where c.u. and c.d.

$h'(x) = 5x^4 - 5 = 5(x^4 - 1) = 5(x^2+1)(x^2-1)$

$h''(x) = 20x^3$

$x > 0,\ h''(x) > 0$; c.u

$x < 0,\ h''(x) < 0$; c.d



rel. extr: CN: $x = \pm 1$

$h''(-1) < 0 \ \therefore$ c.d $(-1, 6)$ rel MAX

$h''(1) > 0 \ \therefore$ c.u $(1, -2)$ rel MIN

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Problems for 3.4

3.4 Concavity & the 2nd Derivate Test

Definition of Point of Inflection

Let f be a function that is continuous on an open interval and let c be a point in the interval. If the graph of f has a tangent line at this point $(c, f(c))$, then this point is a **point of inflection** of the graph of f if the concavity of f changes from upward to downward (or downward to upward) at the point.

Inflection Point at c

1. f continuous
2. f has a tangent line
3. concavity changes
   (f '' changes sign)

If $(c, f(c))$ is a point of inflection of the graph of f, then either $f''(c) = 0$ or $f''(c)$ does not exist at x = c.

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3.4 Concavity & the 2nd Derivate Test

Theorum: Points of Inflection

If $(c, f(c))$ is a point of inflection of the graph of f, then either:

$f''(c) = 0$ or f'' does not exist at $x = c$

how?

Consider: $y = x^{1/3}$ has IP and $y = x^{2/3}$ no IP

Change f(x) to x^(1/3)

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Problems for 3.4

3.4 Concavity & the 2nd Derivate Test

Theorum: Points of Inflection

If $(c, f(c))$ is a point of inflection of the graph of f, then either:

$f''(c) = 0$ or f'' does not exist at $x = c$

how?

$y = x^{1/3}$

$\frac{dy}{dx} = \frac{1}{3}x^{-2/3}$

$\frac{d^2y}{dx^2} = -\frac{2}{9}x^{-5/3} = \frac{-2}{9x^{5/3}}$

Consider:

$y = x^{1/3}$

(0,0) is IP

$\begin{cases} x<0, \ \frac{d^2y}{dx^2} > 0 \ \text{c.d.} \\ x>0, \ \frac{d^2y}{dx^2} < 0 \ \text{c.d.} \end{cases}$

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Problems for 3.4

3.4 Concavity & the 2nd Derivate Test

Example: G: $f(x) = 2x^3 - 3x^2 - 12x + 5$
F: IP

Compare f(x), f '(x), f "(x)

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Problems for 3.4

3.4 Concavity & the 2nd Derivate Test

Example: G: $f(x) = 2x^3 - 3x^2 - 12x + 5$
F: IP $\xrightarrow{\text{cont.}}$ tang. line exists, conc. change. f''
of polyn.

$f'(x) = 6x^2 - 6x - 12 = 6(x^2 - x - 2)$
$= 6(x-2)(x+1)$

$f''(x) = 12x - 6$
$= 6(2x-1)$

poss IP when $f''(x) = 0$

$2x - 1 = 0$
$x = \frac{1}{2}$

	$(-\infty, \frac{1}{2})$	$(\frac{1}{2}, \infty)$
x	0	1
$f'' = 6(2x-1)$	$-$	$+$
$f(x)$	c.d	c.u

$f(\frac{1}{2}) = 2(\frac{1}{2})^3 - 3(\frac{1}{2})^2 - 12(\frac{1}{2}) + 5$
$= \frac{2}{8} - \frac{3}{4} - 6 + 5$
$= \frac{2-6}{8} - 1 = \frac{-4}{8} - 1 = -3/2$

$\therefore (\frac{1}{2}, \frac{-3}{2})$ IP

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Problems for 3.4

3.4 Concavity & the 2nd Derivate Test

Rel. Extr. 2nd Deriv. Test

Example: G: $f(x) = 2x^3 - 3x^2 - 12x + 5$

F: [redacted]

f''

→ cont. tang. line exists, conc. change.

ok polyn.

$f'(x) = 6x^2 - 6x - 12 = 6(x^2 - x - 2)$
$= 6(x-2)(x+1)$

$f''(x) = 12x - 6$
$= 6(2x-1)$

CN $f'(x) = 0$ ~~or DNE~~

$\boxed{x = 2, -1 \text{ CN}}$

$6(x-2)(x+1) = 0$

$(x-2)(x+1) = 0$

$x - 2 = 0 \quad | \quad x + 1 = 0$

$x = 2 \quad | \quad x = -1$

$f''(2) > 0 \therefore$ c.u. $(2, -15)$ rel. MIN

$f''(-1) < 0 \therefore$ c.d. $(-1, 12)$ rel MAX

$x = 2$

$f(2) = 2(2^3) - 3(2^2) - 12(2) + 5$
$16 - 12 - 24 + 5 = -15$
$-20 + 5$

$x = -1$

$f(-1) = 2(-1)^3 - 3(-1)^2 - 12(-1) + 5$
$= -2 - 3 + 12 + 5 = 12$

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