2.3 Product & Quotient Rules of Differentiation

2.3 # 1-7, 11, 15, 17, 21, 22, 23, 24
29, 31, 37, 39, 43, 53, 57, 59
61, 63, 67, 75, 83, 87, 93, 97, 101

Class Notes: Prof. G. Battaly, Westchester Community College, NY

Calculus Home Page Homework Part A Part B

2.3 Product & Quotient Rules of Differentiation

$y = x^3(x^2+1)$ F: $\frac{dy}{dx}$

How do find derivative?

2.3 Product & Quotient Rules of Differentiation

$y = x^3(x^2+1)$ F: $\frac{dy}{dx}$

How do find derivative?

What is the **wrong** approach?

2.3 Product & Quotient Rules of Differentiation

$y = x^3(x^2+1)$ F: $\frac{dy}{dx}$

How do find derivative?

$x^m x^n = x^{m+n}$

$y = x^5 + x^3$ Distributive Property

$\frac{dy}{dx}$ $5x^4 + 3x^2$

2.3 Product & Quotient Rules of Differentiation

$y = (x-1)(x+1)$ F: $\frac{dy}{dx}$

How do find derivative?

$y = x^2 - 1$

$\frac{dy}{dx} = 2x$

2.3 Product & Quotient Rules of Differentiation

$y = \sqrt{x}(x+1)$ F: $\frac{dy}{dx}$

How do find derivative?

$y = \sqrt{x}\,x + \sqrt{x}$

$= x^{1/2}x^{1} + x^{1/2}$

$= x^{3/2} + x^{1/2}$

$\frac{dy}{dx} = \frac{3}{2}x^{1/2} + \frac{1}{2}x^{-1/2} = \frac{3}{2}x^{1/2} + \frac{1}{2x^{1/2}}$

$= \frac{3x^{1/2}}{2}\cdot\frac{x^{1/2}}{x^{1/2}} + \frac{1}{2x^{1/2}}$

$= \frac{3x^{1}+1}{2x^{1/2}} = \frac{3x+1}{2\sqrt{x}}$

$x^m x^n = x^{m+n}$

2.3 Product & Quotient Rules of Differentiation

$y = (x+1)(\sqrt{x+4})$

$y = x \sin x$

What about a product we cannot expand?

We do not have a rule yet to handle this.

$y = x\sqrt{x+4} + \sqrt{x+4}$

2.3 Product & Quotient Rules of Differentiation

Product Rule

If f and g both differentiable

then $\frac{d}{dx}\left[f(x)g(x)\right] = f(x)g'(x) + g(x)f'(x)$

Product Rule

Original problem done with the Product Rule

$y = x^3(x^2+1)$

$\frac{dy}{dx} = x^3(2x) + (x^2+1)(3x^2)$

$= 2x^4 + 3x^4 + 3x^2 = 5x^4 + 3x^2$

2.3 Product & Quotient Rules of Differentiation

$f(x) = (6x+5)(x^3-2)$ Product Rule F: $f'(x)$

$$f'(x) = (6x+5)\frac{d(x^3-2)}{dx} + (x^3-2)\frac{d(6x+5)}{dx}$$

$$= (6x+5)(3x^2) + (x^3-2)(6)$$

$$= 18x^2 + 15x^2 + 6x^3 - 12$$

$$= 24x^3 + 15x^2 - 12$$

2.3 Product & Quotient Rules of Differentiation

F: $g'(x)$

Product Rule

$$g(x) = \sqrt{x}\,(4 - x^2) = x^{1/2}(4 - x^2)$$

$$g'(x) = x^{1/2}(-2x) + (4 - x^2)\left(\tfrac{1}{2}x^{-1/2}\right)$$

$$= -2x^{3/2} + \frac{4 - x^2}{2x^{1/2}}$$

$$= -2x^{3/2}\cdot\frac{2x^{1/2}}{2x^{1/2}} + \frac{4 - x^2}{2x^{1/2}}$$

$$= \frac{-4x^2 + 4 - x^2}{2x^{1/2}} = \frac{4 - 5x^2}{2\sqrt{x}}$$

2.3 Product & Quotient Rules of Differentiation

$y = 2x(4 - x^3)$ F: $\frac{dy}{dx}$

Product Rule

$\frac{dy}{dx} = 2x(-3x^2) + (4 - x^3)(2)$

$= -6x^3 + 8 - 2x^3$

$= -8x^3 + 8$

$-8x^3 + 8$

6. $g(x) = \sqrt{x}\sin x$ F: $g'(x)$

Product Rule

$$g(x) = x^{1/2}\sin x$$

$$g'(x) = x^{1/2}\frac{d(\sin x)}{dx} + \sin x\,\frac{d(x^{1/2})}{dx}$$

$$= x^{1/2}\cos x + \frac{1}{2}x^{-1/2}\sin x$$

$$= x^{1/2}\cos x + \frac{\sin x}{2x^{1/2}}$$

$$g(t) = \frac{t^2+2}{2t-7} \qquad = (t^2+2)(2t-7)^{-1}$$

QUOTIENT RULE

Quotient Rule

The quotient $\frac{f}{g}$, f, g diff.

is diff. at all $x \Rightarrow g(x) \neq 0$

and $\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2}$

$$\frac{D\,dN - N\,dD}{D^2}$$

$$h(t) = \frac{t^2+2}{2t-7}$$

Quotient Rule

QR: $\frac{D\,dN - N\,dD}{D^2}$

$$h'(t) = \frac{(2t-7)\frac{d(t^2+2)}{dt} - (t^2+2)\frac{d(2t-7)}{dt}}{(2t-7)^2}$$

$$= \frac{(2t-7)(2t) - (t^2+2)(2)}{(2t-7)^2}$$

$$= \frac{4t^2-14t-2t^2-4}{(2t-7)^2} = \frac{2t^2-14t-4}{(2t-7)^2}$$

12. $f(t) = \dfrac{\cos t}{t^3} = t^{-3}\cos t$

PR.

$f'(t) = t^{-3}\dfrac{d(\cos t)}{dt} + \cos t\, d(t^{-3})$

$= t^{-3}(-\sin t) + \cos t(-3t^{-4})$

$= \dfrac{(-\sin t)}{t\ t^3} - \dfrac{3\cos t}{t^4}$

$= \dfrac{-t\sin t - 3\cos t}{t^4}$

Quotient Rule

$\dfrac{D\,dN - N\,dD}{D^2}$

$f'(t) = \dfrac{t^3\dfrac{d(\cos t)}{dt} - \cos t\dfrac{d(t^3)}{dt}}{(t^3)^2}$

$= \dfrac{t^3(-\sin t) - 3t^2\cos t}{t^6}$

$= \dfrac{t^2[-t\sin t - 3\cos t]}{t^6}$

$= \dfrac{-t\sin t - 3\cos t}{t^4}$

29. $G: f(x) = \dfrac{2x+5}{\sqrt{x}}$ $F: f'(x)$ Quotient Rule

$f(x) = x^{-1/2}(2x+5) = 2x^{1/2} + 5x^{-1/2}$ Distributive Prop. & Power Rule

$f'(x) = 2\cdot\frac{1}{2}x^{-1/2} + 5\left(-\frac{1}{2}x^{-3/2}\right) = x^{-1/2} - \frac{5}{2}x^{-3/2}$

$= \dfrac{1}{\sqrt{x}} - \dfrac{5}{2x\sqrt{x}} = \dfrac{2x-5}{2x\sqrt{x}}$

$f(x) = (2x+5)\,x^{-1/2}$ Product Rule

$f'(x) = (2x+5)\left(-\frac{1}{2}x^{-3/2}\right) + \left(x^{-1/2}\right)(2)$

$= -\dfrac{(2x+5)}{2x^{3/2}} + \dfrac{2}{x^{1/2}}\cdot\dfrac{2x}{2x} = \dfrac{-2x-5+4x}{2x\sqrt{x}} = \dfrac{2x-5}{2x\sqrt{x}}$

$f(x) = \dfrac{2x+5}{\sqrt{x}} = \dfrac{2x+5}{x^{1/2}}$ Quotient Rule $\dfrac{DdN - NdD}{D^2}$

$f'(x) = \dfrac{x^{1/2}(2) - (2x+5)\left(\frac{1}{2}x^{-1/2}\right)}{\left(x^{1/2}\right)^2}$

$= \dfrac{\left(2x^{1/2} - \dfrac{2x+5}{2x^{1/2}}\right)}{x}\cdot\dfrac{2x^{1/2}}{2x^{1/2}} = \dfrac{4x-(2x+5)}{2x^{3/2}} = \dfrac{2x-5}{2x^{3/2}}$

Class Notes: Prof. G. Battaly, Westchester Community College, NY

Calculus Home Page Homework Part A Part B

47. $y = \frac{3(1-\sin x)}{2\cos x}$ $F: \frac{dy}{dx}$ Quotient Rule $\frac{DdN - NdD}{D^2}$

$y = \frac{3}{2} \cdot \frac{1-\sin x}{\cos x}$

$$\frac{dy}{dx} = \frac{3}{2}\left[\frac{\cos x \frac{d(1-\sin x)}{dx} - (1-\sin x)\frac{d(\cos x)}{dx}}{\cos^2 x}\right]$$

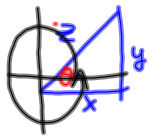
$$= \frac{3}{2}\left[\frac{\cos x(-\cos x) - (1-\sin x)(-\sin x)}{\cos^2 x}\right]$$

$$= \frac{3}{2}\left[\frac{-\cos^2 x + \sin x(1-\sin x)}{\cos^2 x}\right] = \frac{3}{2}\left[\frac{-\cos^2 x + \sin x - \sin^2 x}{\cos^2 x}\right]$$

$$= \frac{3}{2}\left[\frac{-(\sin^2 x + \cos^2 x) + \sin x}{\cos^2 x}\right] = \frac{3}{2}\left[\frac{\sin x - 1}{\cos^2 x}\right]$$

To convert to trig form using sec x and tan x

$$= \frac{3}{2}\left[\frac{1}{\cos x} \cdot \frac{\sin x}{\cos x} - \frac{1}{\cos^2 x}\right] \quad \frac{3}{2}(\sec x \tan x - \sec^2 x)$$



$\sin^2\theta + \cos^2\theta$

$\left(\frac{y}{z}\right)^2 + \left(\frac{x}{z}\right)^2 = \frac{y^2}{z^2} + \frac{x^2}{z^2} = \frac{x^2+y^2}{z^2}$

$= \frac{z^2}{z^2} = 1$

47. p. 123

$\frac{d(\tan x)}{dx} = (\sec x)^2$

$\frac{d(\sec x)}{dx} = \sec x \tan x$

$\frac{d(\cot x)}{dx} = -\csc^2 x$

$\frac{d(\csc x)}{dx} = -\csc x \cot x$

Note: can use $\frac{3}{2}(\sec x - \tan x)$

(no QR needed)

$y = \frac{3(1-\sin x)}{2\cos x}$

$y = \frac{3}{2} \cdot \frac{1-\sin x}{\cos x}$

$y = \frac{3}{2}(\sec x - \tan x)$

$\frac{dy}{dx} = \frac{3}{2}(\sec x \tan x - \sec^2 x)$

$\frac{1-\sin x}{\cos x} = \frac{1}{\cos x} - \frac{\sin x}{\cos x}$

$= \sec x - \tan x$

Lots easier, but need to know your trig!

Distributive Prop. & Power Rule

$= t^{1/2}(1-t^2) = t^{1/2} - t^{5/2}$

$h'(t) = \frac{1}{2}t^{-1/2} - \frac{5}{2}t^{3/2}$

$= \frac{1}{2t^{1/2}} - \frac{5}{2}t^{3/2} \cdot \frac{t^{1/2}}{t^{1/2}}$

3. $h(t) = \sqrt{t}(1-t^2)$

$h(t) = t^{1/2}(1-t^2)$

$h'(t) = t^{1/2}\frac{d(1-t^2)}{dt} + (1-t^2)\frac{d(t^{1/2})}{dt}$

$= t^{1/2}(-2t^1) + (1-t^2)\left(\frac{1}{2}t^{-1/2}\right)$

$-2t^{3/2} + \frac{1-t^2}{2t^{1/2}}$

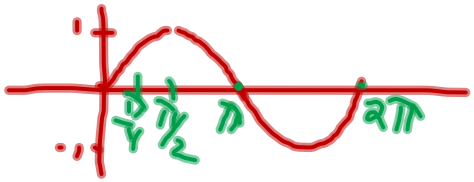
$\frac{2t^{1/2}}{2t^{1/2}}(-2t^{3/2}) + \frac{1-t^2}{2t^{1/2}} = \frac{-4t^2+1-t^2}{2t^{1/2}} = \frac{-5t^2+1}{2t^{1/2}}$

17. $G: f(x) = x\cos x$ $\quad F: f'\left(\frac{\pi}{4}\right)$

$$f'(x) = x\frac{d(\cos x)}{dx} + (\cos x)\frac{d(x)}{dx}$$

$$= x(-\sin x) + \cos x$$

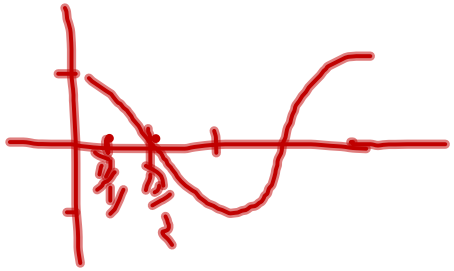
$$= -x\sin x + \cos x$$



$$f'\left(\frac{\pi}{4}\right) = -\frac{\pi}{4}\sin\frac{\pi}{4} + \cos\frac{\pi}{4}$$

$$= -\frac{\pi}{4}\cdot\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}$$

$$= \frac{1}{\sqrt{2}}\left(1 - \frac{\pi}{4}\right)$$



94. G: $f(x) = 8x^6 - 10x^5 + 5x^3$ F: $f''(x)$

$f'(x) = 48x^5 - 50x^4 + 15x^2$

$f''(x) = 240x^4 - 200x^3 + 30x$

27. G: $f(x) = x\left(1 - \frac{4}{x+3}\right)$ F: $f'(x)$

Quotient Rule

$f(x) = x - \frac{4x}{x+3} = \frac{x(x+3) - 4x}{x+3}$

$f'(x) = 1 - \left[\frac{(x+3)(4) - 4x(1)}{(x+3)^2}\right] = 1 - \left[\frac{4x+12-4x}{(x+3)^2}\right]$

$= 1 - \frac{12}{(x+3)^2} = \frac{(x+3)^2 - 12}{(x+3)^2}$

$\frac{x^2+6x+9-12}{(\quad)^2} = \frac{x^2+6x-3}{(\quad)^2}$

$(\quad)(\quad) = -3$

$(\quad)+(\quad) = 6$

$-3 \mid 1 \quad 6 \quad -3$

$\quad\quad\quad -3 \quad -9$

$\quad\quad 1 \quad 3 \mid -12$

94. G: $f(x) = x + 32x^{-2}$ F: $f''(x)$

$f'(x) = 1 + 32(-2x^{-3})$

$= 1 - 64x^{-3}$

$f''(x) = (-64)(-3x^{-4})$

$= 192x^{-4}$

$f'(x)$

$f''(x)$

$f'''(x)$

$y = x^4 + 3x^2 - 1$

F: $\frac{d^2y}{dx^2}$

$\frac{dy}{dx} = 4x^3 + 6x$

$\frac{d^2y}{dx^2} = 12x^2 + 6$