Series and Tests, a Summary

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Series/Test	Form	Converges	Diverges	Sum/Notes
Alternating	$\sum_{n=1}^{\infty}(-1)^n a_n$ or $\sum_{n=1}^{\infty}(-1)^{n+1} a_n$ $a_n > 0$	Converges if: 1. $\lim_{n\to\infty} a_n = 0$ 2. $a_{n+1} \le a_n$ for all n		s_N as approx. with Remainder: $\lvert R_N \rvert \le a_{N+1}$
Direct Comparison	Compare Σa_n to known Σb_n where $a_n, b_n > 0$ $\sum_{n=1}^{\infty} a_n$ $\sum_{n=1}^{\infty} b_n$	If $0 < a_n \le b_n$ & if Σb_n converges, then $\mathbf{\Sigma a_n}$ converges	If $0 < b_n \le a_n$ & if Σb_n diverges, then $\mathbf{\Sigma a_n}$ diverges	
Geometric	$\sum_{n=1}^{\infty} ar^{n-1}$ or $\sum_{n=0}^{\infty} ar^n$	**$\lvert r \rvert < 1$**	**$\lvert r \rvert \ge 1$**	$S = \frac{a}{1-r}$
Integral Test	$\sum_{n=1}^{\infty} a_n$ $a_n = f(n) \ge 0$; $f(n)$ continuous, positive, decreasing	$\int_1^{\infty} f(x)dx$ converges	$\int_1^{\infty} f(x)dx$ diverges	s_N as approx. with Remainder: $0 < R_N \le \int_N^{\infty} f(x)dx$
Limit Comparison	Compare Σa_n to known Σb_n where $a_n, b_n > 0$ and $\lim_{n\to\infty}\left(\frac{a_n}{b_n}\right) = L > 0$	If Σb_n converges, **Σa_n converges**	If Σb_n diverges, **Σa_n diverges**	
nth Term Divergence	$\sum_{n=1}^{\infty} a_n$		$\lim_{n\to\infty} a_n \neq 0$	
p - Series	$\sum_{n=1}^{\infty} \frac{1}{n^p}$	**$p > 1$**	**$0 < p \le 1$**	
Ratio	$\sum_{n=1}^{\infty} a_n$ $a_n \neq 0$	$\lim_{n\to\infty}\left\lvert\frac{a_{n+1}}{a_n}\right\rvert < 1$ converges absolutely	$\lim_{n\to\infty}\left\lvert\frac{a_{n+1}}{a_n}\right\rvert > 1$ Diverges	Note: If limit =1, test is inconclusive; Does not work for p-series
Root	$\sum_{n=1}^{\infty} a_n$	$\lim_{n\to\infty}\sqrt[n]{\lvert a_n \rvert} < 1$ converges absolutely	$\lim_{n\to\infty}\sqrt[n]{\lvert a_n \rvert} > 1$ Diverges	Note: If limit =1, test is inconclusive; Does not work for p-series
Telescoping	a_n, includes a difference (partial fract); most terms of s_n add to 0; eg: $\sum_{n=1}^{\infty}(a_n - a_{n+1})$	$\lim_{n\to\infty} a_n = L$		$S = a_n - L$

A Plan for Evaluating Convergence and Divergence of Series

1. Does a_n approach 0?
   - Yes. Step 2
   - No. Series diverges by *nth Term Divergence*
2. Alternating signs?
   - Yes. Alternating Series (if fails - Ratio Test)
   - No. Step 3
3. Easy to Integrate?
   - Yes. Integral Test
   - No. Step 4
4. Geometric Series? or p - Series?
   - Yes. Geometric or p - Series test
   - No. Step 5
5. Other
   - Direct or Limit Comparison (compare to other series, using dominant terms to decide which series)
   - Ratio or Root Tests (compare terms within the series)

Power Series at c:

$$\sum_{n=0}^{\infty} b_n(x-c)^n = b_0 + b_1(x-c) + b_2(x-c)^2 + \ldots$$

centered at c

Taylor Series for f at c:

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(x-c)^n = f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \ldots + \frac{f^{(n)}(c)}{n!}(x-c)^n + \ldots$$

centered at c

Maclaurin Series for f at c=0:

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \ldots + \frac{f^{(n)}(0)}{n!}x^n + \ldots$$

centered at 0