7.4 Integration using Partial Fractions

Goals:

1. Recognize that rational expressions may need to be simplified to be integrable.
2. Use long division to obtain proper fractions, with the degree of the numerator less than the degree of the denominator.
3. Integrands that are rational expressions may be decomposed to simpler forms that are integrable.

To decompose a rational expression:

a) factor the denominator

b) write the problem as the sum of fractions using the factors as denominators

c) for numerators, use the polynomial form with a degree lower than the degree of the denominator:

A for a linear denominator, Bx+C for a quadratic denom

Study 7.4 # 1-11, 15, 17, 21, 25

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7.4 Integration using Partial Fractions

$$\int \frac{x}{x^2+4x+3}\,dx$$

How do we integrate?

1. u substitution?
   need 2x + 4. missing the 2
2. trig substitution?
   might work if complete square, etc.
3. parts?
   need factors and an integrable part
   that gets less complicated when integrated.

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7.4 Integration using Partial Fractions

Adding simple fractions results in more complex composed fractions

$$\frac{1}{2} + \frac{1}{5} = \frac{\_}{} + \frac{\_}{} = \frac{\_\_}{}$$

$$\frac{3}{x} + \frac{-1}{x+1} = \frac{\qquad + \qquad}{x(x+1)} = \frac{\qquad}{x^2+x}$$

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7.4 Integration using Partial Fractions

$$\int \frac{x}{x^2+4x+3}\,dx$$

To integrate, can decompose the fraction

HOW?

examples:

$$\frac{7}{10} = \frac{\_}{2} + \frac{\_}{5} \ ?$$

$$\frac{2x+3}{x^2+x} = \frac{2x+3}{x(x+1)} = \frac{\_\_}{x} + \frac{\_\_}{x+1} \ ?$$

0 2

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7.4 Integration using Partial Fractions

#1

decompose the fraction

1. Factor denominator

2. Write as sum of fractions
denom are factors of orig denom

$$\frac{2x+3}{x^2+x} = \frac{2x+3}{x(x+1)} = \frac{A}{x} + \frac{B}{x+1}$$

The goal is to build a rational expression using the unknown coefficients and then equate the coefficients to those in the given numerator. This results in a decomposition to the original addends.

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7.4 Integration using Partial Fractions

decompose the fraction

1. Factor denominator

2. Write as sum of fractions
denom are factors of orig denom

3. For numerators, use degree less than degree of denom

$$\frac{2x+3}{x^2+x} = \frac{2x+3}{x(x+1)} = \frac{A}{x} + \frac{B}{x+1}$$

A, B constants

4. Find common denominators and add fractions

$$\frac{A}{x} + \frac{B}{x+1} = \frac{A(x+1) + Bx}{x(x+1)} = \frac{Ax+A+Bx}{x(x+1)}$$

5. Combine like terms and equate numerators.

$$\frac{2x+3}{x^2+x} = \frac{(A+B)x+A}{x(x+1)}$$ and $$2x+3 = (A+B)x+A$$

0 2

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7.4 Integration using Partial Fractions

decompose the fraction   denom are factors of orig denom

1. Factor denominator
2. Write as sum of fractions whose denom are factors of the orig denom
3. For numerators, use degree less than denominator
4. Find common denominators and add fractions
5. Combine like terms and equate numerators.

$$\frac{2x+3}{x^2+x} = \frac{2x+3}{x(x+1)} = \frac{A}{x} + \frac{B}{x+1} = \frac{A(x+1)+Bx}{x(x+1)} = \frac{Ax+A+Bx}{x(x+1)}$$

$$\frac{2x+3}{x^2+x} = \frac{(A+B)x+A}{x(x+1)} \qquad 2x+3 = (A+B)x+A$$

6. Equate unknowns to coefficients.

$$A + B = 2$$
$$A = 3, \quad B = -1$$

$$\frac{2x+3}{x^2+x} = \frac{A}{x} + \frac{B}{x+1} = \frac{3}{x} + \frac{-1}{x+1}$$

0 2

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7.4 Integration using Partial Fractions

#2

$$\int \frac{x}{x^2+4x+3}\,dx$$

To integrate, can decompose the fraction

$$\frac{x}{x^2+4x+3} = \frac{x}{(x+3)(x+1)} = \frac{A}{x+3} + \frac{B}{x+1}$$

0 2

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7.4 Integration using Partial Fractions

$$\int \frac{x}{x^2+4x+3}\,dx$$

To integrate, decompose the fraction

$$\frac{x}{x^2+4x+3} = \frac{x}{(x+3)(x+1)} = \frac{A}{x+3} + \frac{B}{x+1} = \frac{A(x+1)+B(x+3)}{(x+3)(x+1)} = \frac{Ax+A+Bx+3B}{(x+3)(x+1)}$$

$$\frac{x}{x^2+4x+3} = \frac{(A+B)x+(A+3B)}{(x+3)(x+1)}$$

$$x = (A+B)x + (A+3B)$$

$$1x+0 = (A+B)x + (A+3B)$$

$A + B = 1$

$A + 3B = 0$

$2B=-1,\ B=-1/2,\ A=3/2$

$$\frac{x}{x^2+4x+3} = \frac{A}{x+3} + \frac{B}{x+1} = \frac{3}{2(x+3)} + \frac{-1}{2(x+1)}$$

0 2

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7.4 Integration using Partial Fractions

$$\int \frac{x}{x^2+4x+3}\,dx$$

To integrate, decompose the fraction

$$\frac{x}{x^2+4x+3} = \frac{A}{x+3} + \frac{B}{x+1} = \frac{3}{2(x+3)} + \frac{-1}{2(x+1)}$$

$$\int \frac{x}{x^2+4x+3}\,dx = \frac{3}{2}\int \frac{dx}{x+3} - \frac{1}{2}\int \frac{dx}{x+1}$$

$$= \frac{3}{2}\ln|x+3| - \frac{1}{2}\ln|x+1| + c$$

0 2

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7.4 Integration using Partial Fractions

#3

$$\int \frac{x-1}{x^3(x+1)}\,dx \qquad \frac{x-1}{x^3(x+1)} = \frac{A}{x+1} + \frac{B}{x} + \frac{C}{x^2} + \frac{D}{x^3}$$

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7.4 Integration using Partial Fractions

#3

$$\int \frac{x-1}{x^3(x+1)}\,dx \qquad \frac{x-1}{x^3(x+1)} = \frac{A}{x+1} + \frac{B}{x} + \frac{C}{x^2} + \frac{D}{x^3}$$

Do we really need x, x^2, and x^3 ?

$$x - 1 = Ax^3 + Bx^2(x+1) + Cx(x+1) + D(x+1)$$

$$x - 1 = Ax^3 + Bx^3+Bx^2 + Cx^2+Cx + Dx+D$$

$$x - 1 = (A+B)x^3 + (B+C)x^2 + (C+D)x+D$$

$A + B$	$= 0$	$A = 2$
$B + C$	$= 0$	$B = -2$
$C + D$	$= 1$	$C = 2$
D	$= -1$	$D = -1$

$$\frac{x-1}{x^3(x+1)} = \frac{2}{x+1} + \frac{-2}{x} + \frac{2}{x^2} + \frac{-1}{x^3}$$

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7.4 Integration using Partial Fractions

$$\int \frac{x-1}{x^3(x+1)}\,dx \qquad \frac{x-1}{x^3(x+1)} = \frac{A}{x+1} + \frac{B}{x} + \frac{C}{x^2} + \frac{D}{x^3}$$

$$\int \frac{x-1}{x^3(x+1)} = \int \frac{2}{x+1} + \int \frac{-2}{x} + \int \frac{2}{x^2} + \int \frac{-1}{x^3}$$

$$\int \frac{x-1}{x^3(x+1)} = 2\ln|x+1| - 2\ln|x| - \frac{2}{x} + \frac{1}{2x^2} + c$$

0 2

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7.4 Integration using Partial Fractions

$$\int \frac{x-1}{x^3(x+1)}\,dx$$

Do we really need x, x^2, and x^3 ?

$$\frac{x-1}{x^3(x+1)} = \frac{A}{x+1} + \frac{B}{x^3}$$

$x - 1 = Ax^3 + B(x+1)$

$x - 1 = Ax^3 + Bx + B$

$A = 0$

$B = 1$

$B = -1$

contradiction!

0 2

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7.4 Integration using Partial Fractions

#4

$$\int \frac{2x-3}{(x-1)^2}\,dx$$

$$\frac{2x-3}{(x-1)^2} = \_\_\_ + \_\_\_\_\_$$

0

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7.4 Integration using Partial Fractions

$$\int \frac{2x-3}{(x-1)^2}\,dx$$

$$\frac{2x-3}{(x-1)^2} = \frac{A}{x-1} + \frac{B}{(x-1)^2}$$

$$2x-3 = A(x-1) + B = Ax - A + B$$

$$A = 2$$

$$B-A = -3, \quad B = -1$$

$$\int \frac{2x-3}{(x-1)^2} = \int \frac{2}{x-1}\,dx + \int \frac{-1}{(x-1)^2}\,dx$$

$$= 2\ln|x-1| + \frac{1}{x-1} + c$$

0

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7.4 Integration using Partial Fractions

#5

$$\int \frac{8x^3+13x}{(x^2+2)^2}\,dx$$

$$\frac{8x^3+13x}{(x^2+2)^2} = \frac{Ax+B}{x^2+2} + \frac{Cx+D}{(x^2+2)^2}$$

3. For numerators, use **degree less than degree of the denominator**: Denominator is quadratic or power of quadratic, so use a linear form in the numerator.

0 2

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7.4 Integration using Partial Fractions

$$\int \frac{8x^3+13x}{(x^2+2)^2}\,dx$$

$$\frac{8x^3+13x}{(x^2+2)^2} = \frac{Ax+B}{x^2+2} + \frac{Cx+D}{(x^2+2)^2} = \frac{(Ax+B)(x^2+2) + Cx+D}{(x^2+2)^2}$$

$$Ax^3+2Ax+Bx^2+2B+Cx+D = Ax^3+Bx^2+(2A+C)x+(2B+D)$$

$$8x^3+13x = Ax^3+Bx^2+(2A+C)x+(2B+D)$$

A			= 8	A=8
	B		= 0	B=0
2A		+C	= 13	C=-3
	2B	+D	= 0	D=0

$$\int \frac{8x^3+13x}{(x^2+2)^2}\,dx = \int \frac{8x}{x^2+2}\,dx + \int \frac{-3x}{(x^2+2)^2}\,dx$$

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7.4 Integration using Partial Fractions

$$\int \frac{8x^3+13x}{(x^2+2)^2}\,dx = \int \frac{8x}{x^2+2}\,dx + \int \frac{-3x}{(x^2+2)^2}\,dx$$

$$\int \frac{8x^3+13x}{(x^2+2)^2}\,dx = \frac{8}{2}\int \frac{2x}{x^2+2}\,dx + \frac{-3}{2}\int \frac{2x}{(x^2+2)^2}\,dx$$

$u=x^2+2$
$du=2x\,dx$

$$4\int \frac{du}{u} - \frac{3}{2}\int u^{-2}\,du$$

$$4\ln|u| - \frac{3}{2}\,\frac{u^{-1}}{-1} + c$$

$$4\ln|x^2+2| + \frac{3}{2}\,\frac{1}{x^2+2} + c$$

2
0 2 2

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7.4 Integration using Partial Fractions

#6

$$\int \frac{x^2+x+1}{x^3+x}\,dx$$

0 2

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7.4 Integration using Partial Fractions

$$\int \frac{x^2+x+1}{x^3+x}\,dx$$

To integrate, decompose the fraction

$$\frac{x^2+x+1}{x(x^2+1)} = \frac{A}{x} + \frac{Bx+C}{x^2+1} = \frac{A(x^2+1)+(Bx+C)x}{x(x^2+1)} = \frac{Ax^2+A+Bx^2+Cx}{x(x^2+1)}$$

$$\frac{x^2+x+1}{x(x^2+1)} = \frac{(A+B)x^2+Cx+A}{x(x^2+1)}$$

$$x^2+x+1 = (A+B)x^2+Cx+A$$

$A + B = 1$ B=0
$C = 1$ C=1
$A = 1$ A=1

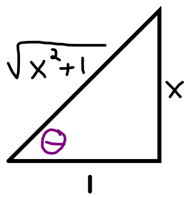
$$\frac{x^2+x+1}{x(x^2+1)} = \frac{A}{x} + \frac{Bx+C}{x^2+1} = \frac{1}{x} + \frac{1}{x^2+1}$$

$$\int \frac{x^2+x+1}{x^3+x}\,dx = \int \frac{1}{x}\,dx + \int \frac{1}{x^2+1}\,dx$$

0 2

7.4 Integration using Partial Fractions

$$\int \frac{x^2+x+1}{x^3+x}\,dx = \int \frac{1}{x}\,dx + \int \frac{1}{x^2+1}\,dx$$



$$= \ln|x| + \int \frac{\sec^2\theta\,d\theta}{\sec^2\theta}$$

$\tan\theta = x$
$\sec^2\theta\,d\theta = dx$
$\sec\theta = \sqrt{x^2+1}$
$(\sec\theta)^2 = (\sqrt{x^2+1})^2$

$$= \ln|x| + \int d\theta$$

$$= \ln|x| + \theta + c$$

$$= \ln|x| + \arctan x + c$$

0 2

7.4 Integration using Partial Fractions

$$\int \frac{x}{16x^4-1}\,dx$$

7.4 Integration using Partial Fractions

$$\int \frac{x}{16x^4-1}\,dx$$

$$\frac{x}{(4x^2+1)(4x^2-1)} = \frac{x}{(4x^2+1)(2x+1)(2x-1)} = \frac{Ax+B}{4x^2+1} + \frac{C}{2x+1} + \frac{D}{2x-1}$$

$$x = (Ax+B)(4x^2-1)+C(4x^2+1)(2x-1)+D(4x^2+1)(2x+1)$$

$$x = 4Ax^3-Ax+4Bx^2-B+C(8x^3-4x^2+2x-1)+D(8x^3+4x^2+2x+1)$$

$$x = (4A+8C+8D)x^3+(4B-4C+4D)x^2+(-A+2C+2D)x+(-B-C+D)$$

$4A + 8C + 8D = 0$	$A=-1/2$
$4B - 4C + 4D = 0$	$B=0$
$-A + 2C + 2D = 1$	$C=1/8$
$-B - C + D = 0$	$D=1/8$

Use matrix algebra on calculator.

$$\frac{x}{(4x^2+1)(2x+1)(2x-1)} = \frac{-x}{2(4x^2+1)} + \frac{1}{8(2x+1)} + \frac{1}{8(2x-1)}$$

$$\int \frac{x}{(4x^2+1)(2x+1)(2x-1)} = \int \frac{-x\,dx}{2(4x^2+1)} + \int \frac{1\,dx}{8(2x+1)} + \int \frac{1\,dx}{8(2x-1)}$$

$$\frac{-1}{16}\ln(4x^2+1) + \frac{1}{16}\ln|2x+1| + \frac{1}{16}\ln|2x-1| + c$$

7.4 Integration using Partial Fractions

$$\int \frac{x}{16x^4-1}\,dx$$

Different approach to finding constants.

$$\frac{x}{(4x^2+1)(4x^2-1)} = \frac{x}{(4x^2+1)(2x+1)(2x-1)} = \frac{Ax+B}{4x^2+1} + \frac{C}{2x+1} + \frac{D}{2x-1}$$

$$x = (Ax+B)(4x^2-1)+C(4x^2+1)(2x-1)+D(4x^2+1)(2x+1)$$

x=1/2: $1/2 = (Ax+B)(0)+C(4x^2+1)(0)+D(2)(2) = 4D$; **D = 1/8**

x=-1/2: $-1/2 = (Ax+B)(0)+C(2)(-2)+D(2)(0) = -4C$; **C = 1/8**

x=1: $1 = (A+B)(3)+C(5)(1)+D(5)(3) = 3A + 3B + 5C + 15D$

x=2: $2 = (2A+B)(15)+C(17)(3)+D(17)(5) = 30A + 15B + 51C + 85D$

$3A + 3B + 5/8 + 15/8 = 1$ $\quad$ $3A + 3B = -12/8 = -3/2$

$30A+15B + 51/8 + 85/8 = 2$ $\quad$ $30A+15B = -15$

B=0, A=-1/2

$$\frac{x}{(4x^2+1)(2x+1)(2x-1)} = \frac{-x}{2(4x^2+1)} + \frac{1}{8(2x+1)} + \frac{1}{8(2x-1)}$$

$$\int \frac{x}{(4x^2+1)(2x+1)(2x-1)} = \int \frac{-x\,dx}{2(4x^2+1)} + \int \frac{1\,dx}{8(2x+1)} + \int \frac{1\,dx}{8(2x-1)}$$

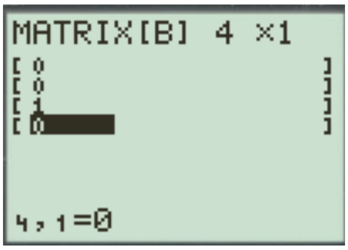
$$\frac{-1}{16}\ln(4x^2+1) + \frac{1}{16}\ln|2x+1| + \frac{1}{16}\ln|2x-1| + c$$

7.4 Integration using Partial Fractions

Matrix Algebra on Calculator to find Constants

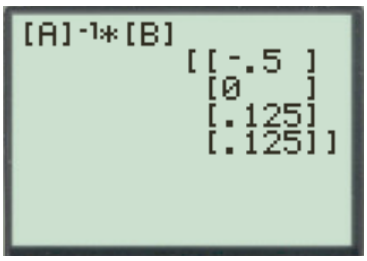
Line up the coefficients in array.

Enter coefficients in matrix A. Enter values in matrix B.



Enter the following operation:

$[A]^{-1} * [B]$ using the inverse button, x^{-1}, and enter



0.125 = 1/8

return to problem