6.5 Mean Value Theorum &Average Value

Goals:
1. Recognize and understand the Mean Value Theorem for Integrals.
2. Find the average value of a function on [a,b].

Study 6.5 # 1-9, 13, 17, 21

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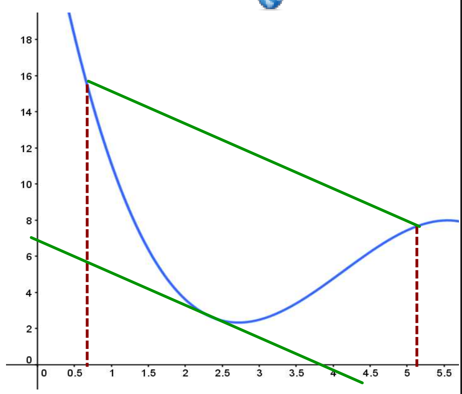
3.2 Rolle's Theorum and the Mean Value Theorum

geogebra demo

Mean Value Theorum for $\boxed{\text{Derivatives}}$



Let f be:
1. continuous on closed interval [a,b] and
2. differentiable on open interval (a,b)

then ∃ at least one c ∈ (a,b) ∋

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

Interpretation:

There exists at least one *c* on the interval from *a* to *b* such that the derivative at *c* equals the slope of the secant line joining the endpoints.

ALSO: There exists at least one c on the interval where the instantaneous rate of change equals the average value.

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Problems for 3.2

6.5 Mean Value Theorum &Average Value

Mean Value Theorum for Integrals

If the MVT for derivatives tells us something about the slope of the tangent line to a curve,

what do you expect the MVT for integrals to tell us?

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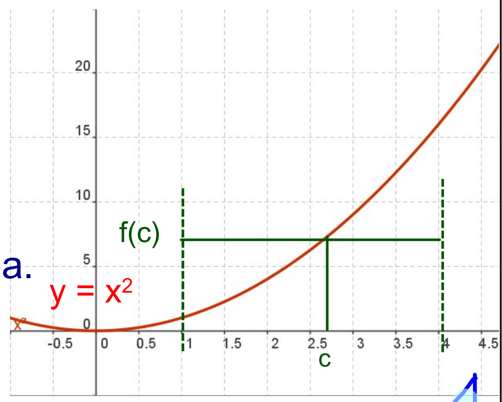
6.5 Mean Value Theorum &Average Value

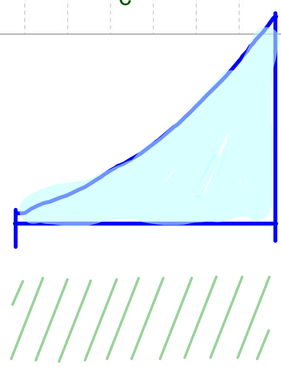
Mean Value Theorum for Integrals

The major interpretation of integrals is area;
MVT for Integrals is about area.



Let f be continuous on a closed interval [a,b]
then ∃ at least one c ∈ [a,b] ∋

$$\int_a^b f(x)dx = f(c)(b - a)$$



(equal areas: area of rectangle, f(c)*(b-a)= area under curve from a to b)

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6.5 Mean Value Theorum &Average Value

Mean Value Theorum for Integrals

Let f be continuous on a closed interval [a,b]
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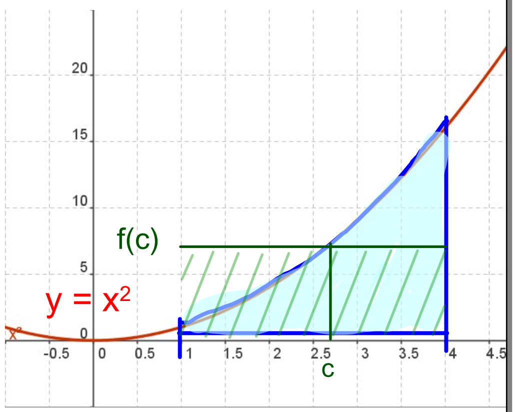
$$\int_a^b f(x)dx = f(c)(b - a)$$



$$\int_1^4 x^2\,dx = x^3/3\Big]_1^4 = 4^3/3 - 1/3 = 63/3 = 21$$

MVT guarantees a value of x, c, so that $f(c)(4-1) = 21$
or $f(c) = 7$

To find c: $x^2 = 7$
$x = \pm\sqrt{7}$
Use $c = \sqrt{7}$ or 2.65

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6.5 Mean Value Theorum &Average Value

Mean Value Theorum for Integrals

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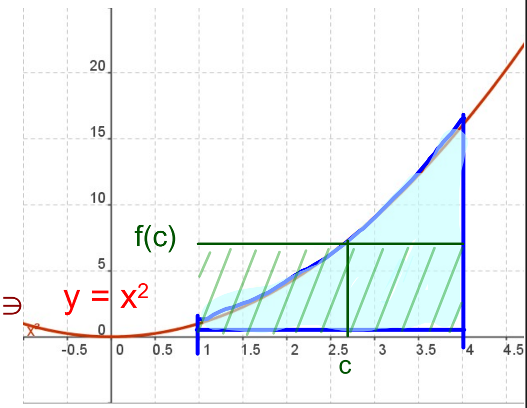

$$\int_1^4 x^2\,dx = 21 = (2.65)^2(4 - 1)$$

(equal areas: area of rectangle, f(c)*(b-a)= area under curve from a to b)

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6.5 Mean Value Theorum &Average Value

Average Value of a Function on [a,b]

From MVT: $\int_a^b f(x)dx = f(c)(b - a)$

Multiply both members of the equation by 1/(b-a)

$$\frac{1}{b-a}\int_a^b f(x)dx = \frac{f(c)(b-a)}{b-a}$$

Results in an equation for **f(c), the average value of the function f(x)** on the interval [a,b]

$$AveVal = f(c) = \frac{1}{b-a}\int_a^b f(x)\,dx$$

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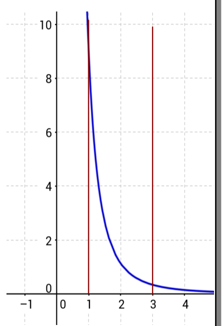
6.5 Mean Value Theorum &Average Value

G: $f(x) = \frac{9}{x^3}$ F: $c \in [1,3] \ni$ MVT applies



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6.5 Mean Value Theorum &Average Value

G: $f(x) = \dfrac{9}{x^3}$ F: c [1,3] MVT applies

f(x) cont on [1,3] $\therefore$ MVT applies $\int_a^b f(x)dx = f(c)(b-a)$

$$\int_1^3 9x^{-3}dx = 9\left[\frac{x^{-2}}{-2}\right]_1^3$$

$$\left[\frac{-9}{2x^2}\right]_1^3 = \frac{-9}{2\cdot 3^2} - \left[-\frac{9}{2\cdot 1^2}\right]$$

$$-\frac{1}{2} + \frac{9}{2} = \frac{8}{2} = 4$$

$$\int_a^b f(x)dx = f(c)(b-a)$$

$$\boxed{\int_1^3 9x^{-3}dx} = f(c)(3-1)$$

$$4 = f(c)(3-1)$$

$$f(c) = 2 = \frac{9}{c^3}$$

$$c^3 = \frac{9}{2} \quad c = \sqrt[3]{\frac{9}{2}} \approx 1.65$$

$\in [1,3]$

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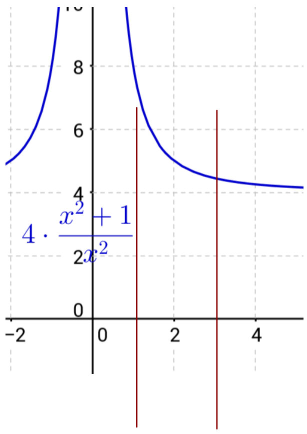
6.5 Mean Value Theorum &Average Value

$$\text{AveVal} = \frac{1}{b-a}\int_a^b f(x)\,dx$$

G: $f(x) = \dfrac{4(x^2+1)}{x^2}$ F: Ave. Value on [1,3]



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6.5 Mean Value Theorum &Average Value

G: $f(x) = \dfrac{4(x^2+1)}{x^2}$ F: Ave. Value on [1,3]

$$\text{AveVal} = \frac{1}{b-a}\int_a^b f(x)\,dx$$

$$Av = \frac{1}{3-1}\int_1^3 \frac{4(x^2+1)}{x^2}\,dx$$

$$\frac{x^2+1}{x^2} = \frac{x^2}{x^2} + \frac{1}{x^2} = 1 + x^{-2}$$

$$= 2\int_1^3 (1+x^{-2})\,dx$$

$$= 2\left[x + \frac{x^{-1}}{-1}\right]_1^3 = 2\left[x - \frac{1}{x}\right]_1^3 = 2\left[\left(3-\frac{1}{3}\right) - (1-1)\right]$$

$$= 2\left[3 - \frac{1}{3}\right] = 2\left(\frac{8}{3}\right) = \frac{16}{3}$$

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6.5 Mean Value Theorum &Average Value

G: $f(x) = \cos x$ F: Ave. Value on $[0,\pi/2]$

$$\text{AveVal} = \frac{1}{b-a}\int_a^b f(x)\,dx$$

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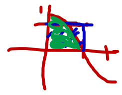
6.5 Mean Value Theorum &Average Value

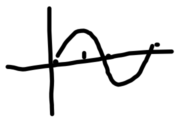


G: $f(x) = \cos x$ F: Ave. Value on $[0,\pi/2]$

$$\text{AveVal} = \frac{1}{b-a}\int_a^b f(x)\,dx$$

$$\text{Ave} = \frac{1}{\pi/2 - 0}\int_0^{\pi/2} \cos x\,dx$$

$$= \frac{2}{\pi}\sin x\Big]_0^{\pi/2} = \frac{2}{\pi}\Big[\overset{1}{\sin\tfrac{\pi}{2}} - \overset{0}{\sin 0}\Big] = \frac{2}{\pi}$$



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6.5 Mean Value Theorum &Average Value

The concentration of a certain drug during the first 20 hrs after it has been administered can be approximated by:

$$p(x) = \frac{300x}{6x^2 + 5}, \quad 0 \le x \le 20$$

where x is the time in hours after the medication is taken and p(x) is the concentration in percent. Determine the average concentration during the first 10 hrs after the medication was taken.

$$\text{AveVal} = \frac{1}{b-a}\int_a^b f(x)\,dx$$

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6.5 Mean Value Theorum &Average Value

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where x is the time in hours after the medication is taken and p(x) is the concentration in percent. Determine the average concentration during the first 10 hrs after the medication was taken.

$$\text{AveVal} = \frac{1}{b-a}\int_a^b f(x)\,dx \qquad a=0,\ b=10$$

$$\text{AveVal} = \frac{1}{10-0}\int_0^{10} \frac{300x}{6x^2+5}\,dx$$

$$\text{AveVal} = \frac{30}{12}\int_0^{10} \frac{12x}{6x^2+5}\,dx$$

$$= \frac{5}{2}\ln|6x^2+5|\Big]_0^{10}$$

$$= \frac{5}{2}[\ln 605 - \ln 5] = (5/2)\ln(605/5)$$

$$= (5/2)\ln(121) = 11.99\%$$

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6.5 Mean Value Theorum &Average Value

F: Average value

$f(x)=\sqrt{x}$ on $[0,4]$

$g(t) = \dfrac{t}{\sqrt{3+t^2}}$ on $[1,3]$

$f(x) = \dfrac{x^2}{(3+x^3)^2}$ on $[-1,1]$

$h(x) = \dfrac{\ln x}{x}$ on $[1,5]$

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6.5 Mean Value Theorum &Average Value

F: Average value

f(x)=√x on [0,4]

G: $f(x)=\sqrt{x}$ F: Ave. value on [0,4]

$$\text{Ave. value} = \frac{1}{b-a}\int_a^b f(x)\,dx$$

$$= \frac{1}{4-0}\int_0^4 x^{1/2}\,dx = \frac{1}{4}\,\frac{x^{3/2}}{3/2}\Big]_0^4$$

$$= \frac{1}{6}x^{3/2}\Big]_0^4 = \frac{8}{6} - 0 = \frac{4}{3}$$

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6.5 Mean Value Theorum &Average Value

F: Average value

$g(t) = \dfrac{t}{\sqrt{3+t^2}}$ on [1,3]

G: $g(t) = \dfrac{t}{\sqrt{3+t^2}}$ F: Ave. value on [1,3]

$$\text{Ave. Val} = \frac{1}{b-a}\int_a^b g(t)\,dt$$

$$= \frac{1}{3-1}\int_1^3 \frac{t}{\sqrt{3+t^2}}\,dt = \frac{1}{2}\frac{1}{2}\int_1^3 (3+t^2)^{-1/2}\,2t\,dt$$

$$= \frac{1}{4}\,\frac{(3+t^2)^{1/2}}{1/2}\Big]_1^3 = \frac{1}{2}\left[\sqrt{12} - \sqrt{4}\right]$$

$$= \frac{1}{2}\left[2\sqrt{3} - 2\right] = \sqrt{3} - 1$$

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6.5 Mean Value Theorum &Average Value

F: Average value

$f(x) = \dfrac{x^2}{(3+x^3)^2}$ on [-1,1]

G: $f(x) = \dfrac{x^2}{(x^3+3)^2}$ F: Ave.val on $[-1,1]$ $x \neq -\sqrt[3]{3}$

$f(x)$ cont, differentiable on $[-1,1]$

$\text{Ave.val} = \frac{1}{b-a}\int_a^b f(x)\,dx$

$= \frac{1}{1-(-1)} \cdot \frac{1}{3}\int_{-1}^{1} (x^3+3)^{-2}\, 3x^2\,dx = \frac{-1}{6}(x^3+3)^{-1}\Big]_{-1}^{1}$

$= -\frac{1}{6(x^3+3)}\Big]_{-1}^{1} = \frac{-1}{6(4)} - \left(\frac{-1}{6(2)}\right)$

$= -\frac{1}{24} + \frac{1}{12} = \boxed{\frac{1}{24}}$

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6.5 Mean Value Theorum &Average Value

F: Average value

$h(x) = \dfrac{\ln x}{x}$ on [1,5]

G: $h(u) = \dfrac{\ln u}{u}$ F: Ave val on $[1,5]$

D: $u \geq 0$

$\text{Ave.Val} = \frac{1}{b-a}\int_a^b h(u)\,du$

$\text{Ave Val} = \frac{1}{5-1}\int_1^5 \frac{\ln u}{u}\,du$ $z = \ln u$, $dz = \frac{1}{u}\,du$

u	z
1	0
5	ln5

$= \frac{1}{4}\int_0^{\ln 5} z\,dz$

$= \frac{1}{4}\,\frac{z^2}{2}\Big]_0^{\ln 5} = \frac{1}{8}(\ln 5)^2 - 0 = \frac{1}{8}(\ln 5)^2$

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