6.1 Area between Two Curves

Goals:

1. Remember that the area under a curve is the sum of the areas of an infinite number of rectangles $A = \lim_{n\to\infty} \sum_{i=1}^{n} f(c_i)\Delta x$
2. Understand the approach to finding the area between 2 curves.
3. Be able to identify the region bounded by the given conditions.
4. Determine the best orientation for the reference rectangle in the region - horizontal or vertical.
5. Set up the definite integral for finding the area.

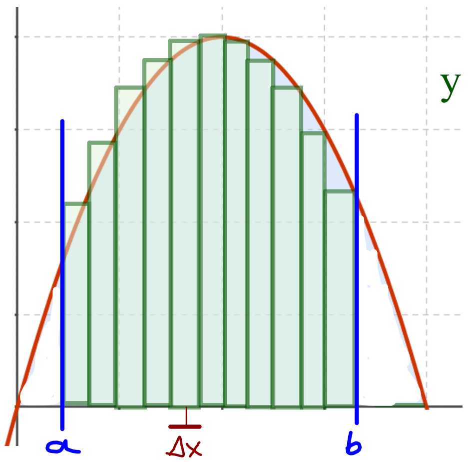
Homework: Study 6.1 # 1, 5, 7, 13, 25, 19; 3, 17, 27, 53

6.1 Area between Two Curves

Background:

Area Under a Curve

To find the **area under the curve** over a specified interval, we would need to divide it into lots of little polygons, such as rectangles, for which we can find the area with known formulas.



$y = -x(x-4)$

$\Delta x = \frac{b-a}{n}$

where n is the number of rectangles

as $n \to \infty$
$\Delta x \to 0$

Areas of small rectangles | $\Delta x\, f(a)$ $\Delta x\, f(a+\Delta x)$ $\Delta x\, f(a+2\Delta x)$ $\Delta x\, f(a+3\Delta x)$

6.1 Area between Two Curves

Background:

Area Under a Curve

Definition of Area of a Region in a Plane

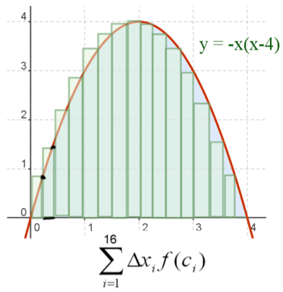
Let [1] f be continuous and [2] non-negative on [a,b].
The area of the region bounded by the graph of f, the x-axis, and the vertical lines x = a and x = b is:

$$A = \lim_{n\to\infty} \sum_{i=1}^{n} f(c_i)\Delta x$$

$x_{i-1} < c_i < x_i$ $\Delta x = \dfrac{b-a}{n}$

as $n \to \infty$, $\Delta x \to 0$

Using 16 rectangles:



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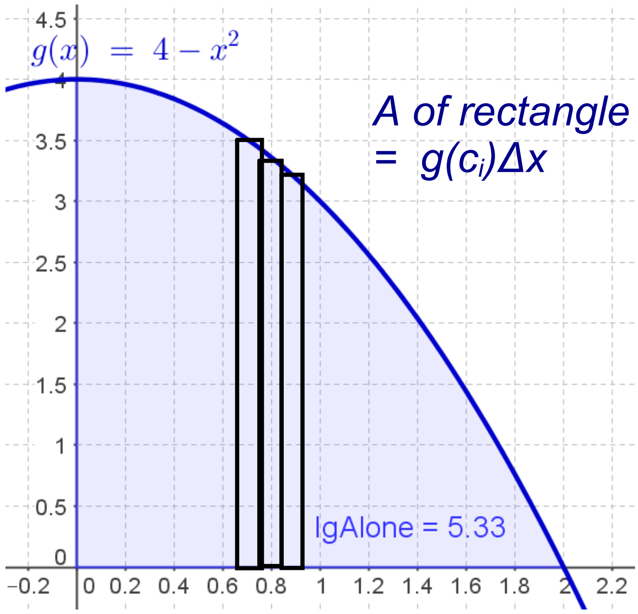
6.1 Area between Two Curves

F: Area of shaded region



$$A = \lim_{n\to\infty} \sum_{i=1}^{n} f(c_i)\Delta x$$

$$\int_a^b f(x)dx = F(b)-F(a)$$

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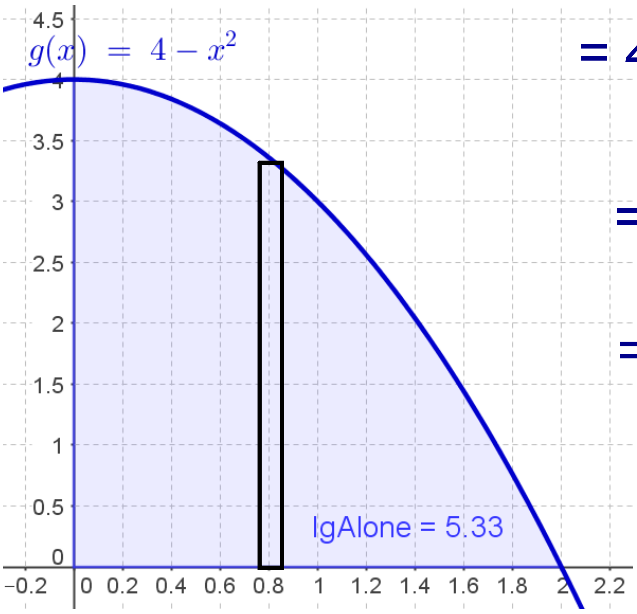
6.1 Area between Two Curves



$$A = \int_0^2 (4-x^2)\, dx$$

$$= 4x - \frac{x^3}{3}\Big]_0^2$$

$$= 4(2) - \frac{8}{3} - 0$$

$$= 8 - \frac{8}{3} = \frac{16}{3}$$

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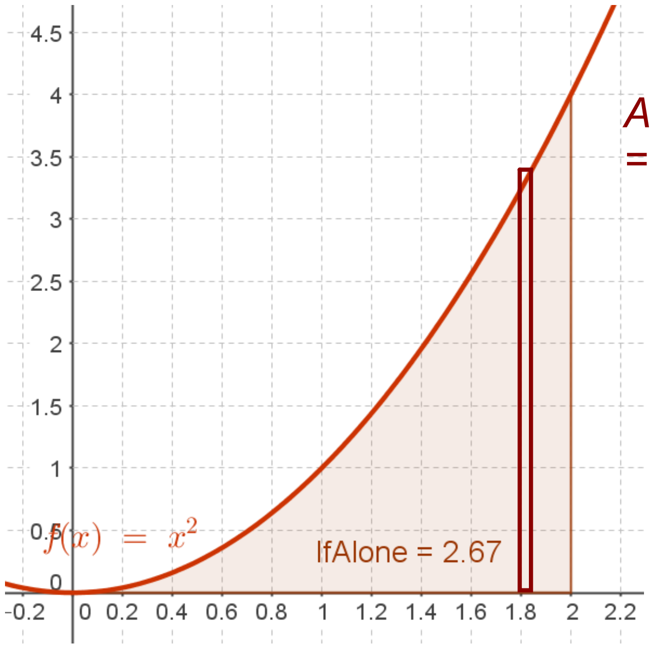
6.1 Area between Two Curves

F: Area of shaded region



A of rectangle
$= f(c_i)\Delta x$

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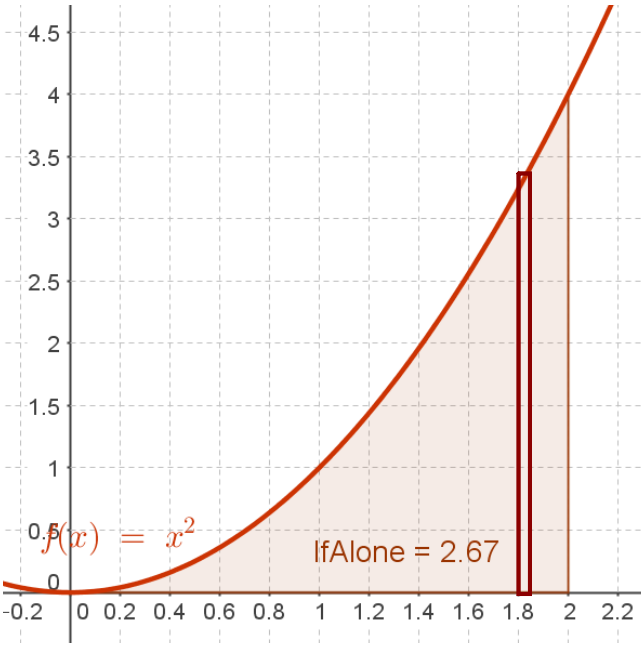
6.1 Area between Two Curves

$$A = \int_0^2 x^2\,dx$$

$$= \frac{x^3}{3}\Big]_0^2$$

$$= \frac{8}{3} - 0$$

$$= \frac{8}{3}$$



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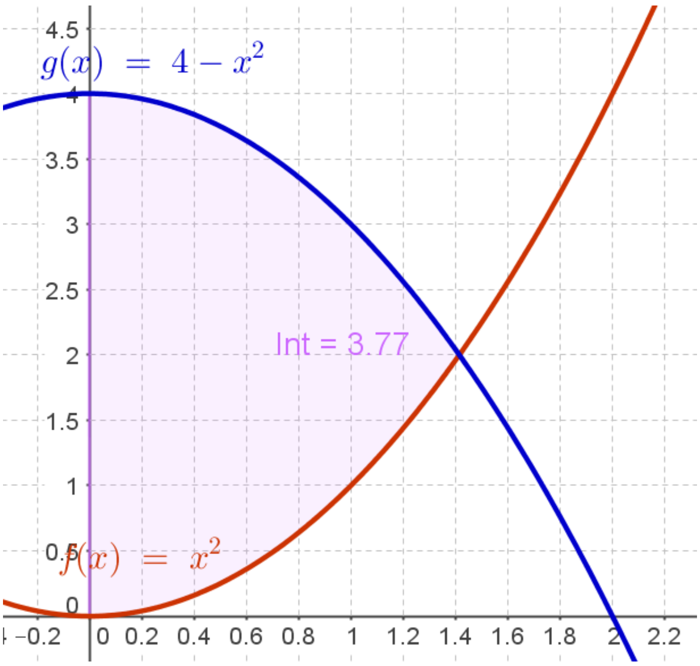
6.1 Area between Two Curves

How to find Area of shaded region?



1st need to find point of intersection, because region does not extend beyond that point.

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6.1 Area between Two Curves

How to find Area of shaded region?

1st need to find point of intersection, because region does not extend beyond that point.

$x^2 = 4 - x^2$

$2x^2 = 4$

$x^2 = 2$

$x = \pm\sqrt{2}$

$\therefore$ UL, $x = +\sqrt{2}$

LL, $x = 0$

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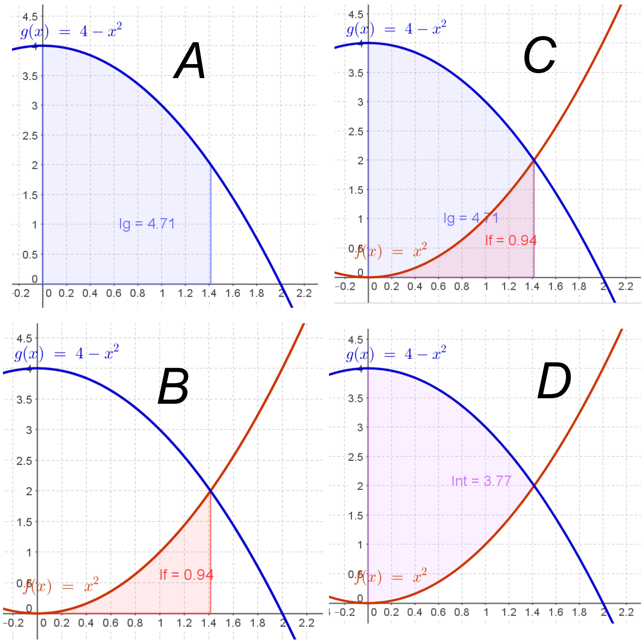
6.1 Area between Two Curves

How to find Area of shaded region?



area betw 2 curves

C shows both A and B together

D shows A - B

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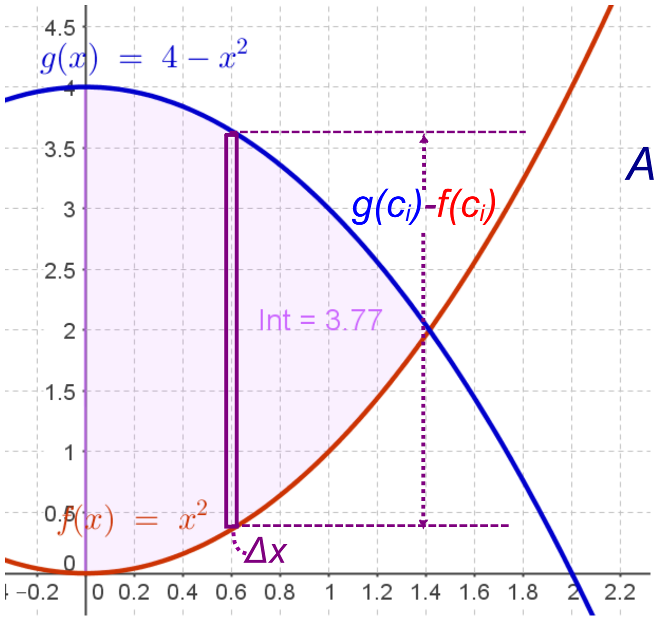
6.1 Area between Two Curves

How to find Area of shaded region?

How do we do that analytically? Use reference rectangle.



A of rectangle $= [g(c_i)-f(c_i)]\Delta x$

$$A = \int_0^{\sqrt{2}} (4-x^2 - x^2)\, dx$$

$$\int_0^{\sqrt{2}} (4-2x^2)\, dx$$

$$= 4x - \frac{2x^3}{3}\Big]_0^{\sqrt{2}}$$

$$= 4\sqrt{2} - \frac{4\sqrt{2}}{3}$$

$$= \frac{8\sqrt{2}}{3}$$

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6.1 Area between Two Curves

Area of region Bounded by 2 Curves

If f and g are continuous on [a, b] and g(x) ≤ f(x) for all x on [a, b], then the area of the region bounded by f and g and the vertical lines and x = a and x = b is:

$$A = \int_a^b [\, f(x) - g(x)\,]\, dx$$

In general, the area between 2 curves that intersect and switch y positions is given by:

$$A = \int_a^b |\, f(x) - g(x)\,|\, dx$$ *for vertical reference rectangle*

or $$A = \int_c^d |\, f(y) - g(y)\,|\, dy$$ *for horizontal reference rectangle*

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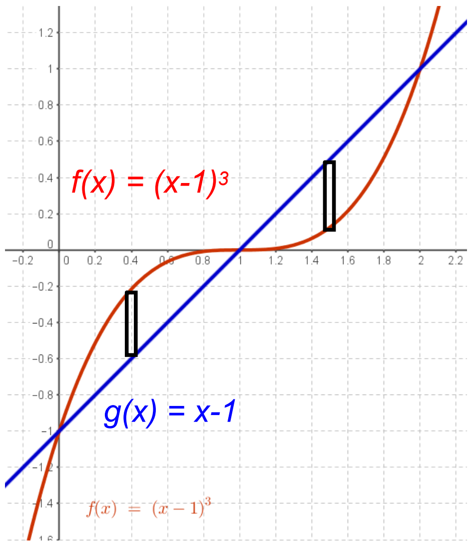
6.1 Area between Two Curves

G: $f(x) = (x-1)^3$, $g(x) = x-1$

F: area of bounded region



6.1 Area between Two Curves

G: $f(x) = (x-1)^3$, $g(x) = x-1$

F: area of bounded region

$$A = \int_0^1 (f-g)dx + \int_1^2 (g-f)dx$$

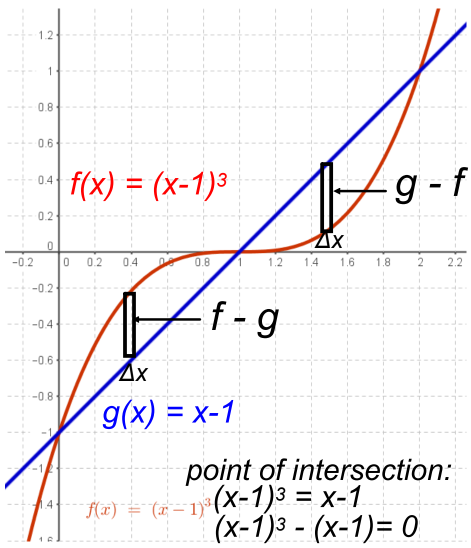
$$A = \tfrac{1}{2}\int_0^1 2(x-1)(x^2-2x)dx + \tfrac{1}{2}\int_1^2 -2(x-1)(x^2-2x)dx$$

$u = x^2-2x$
$du=(2x-2)dx=2(x-1)dx$

$$= \tfrac{1}{2}\frac{(x^2-2x)^2}{2}\Big]_0^1 - \tfrac{1}{2}\frac{(x^2-2x)^2}{2}\Big]_1^2$$

$$= \tfrac{1}{4}(1-2)^2 - 0 \quad -[\tfrac{1}{4}(4-4)^2 - \tfrac{1}{4}(1-2)^2]$$

$$= +\tfrac{1}{4} - [\,0 - \tfrac{1}{4}\,] = \tfrac{1}{4} + \tfrac{1}{4} = \tfrac{1}{2}$$

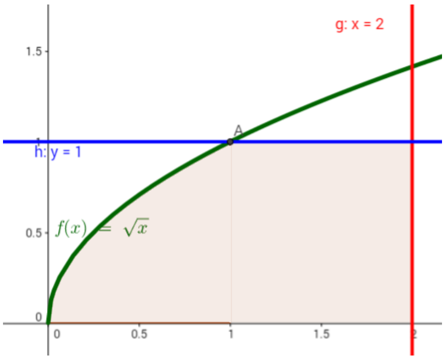


point of intersection:
$(x-1)^3 = x-1$
$(x-1)^3 - (x-1)= 0$
$(x-1)\,[(x-1)^2-1]=0$
$(x-1)\,[(x^2-2x+1-1]=0$
$(x-1)\,(x^2-2x)=0$
$(x-1)(x)(x-2)=0$
$x=0,1,2$

check on calculator:
$A=\int_a^b |\, f-g \,|dx= \int_0^2 \text{abs}[(x-1)^3-(x-1)]dx = 0.5$

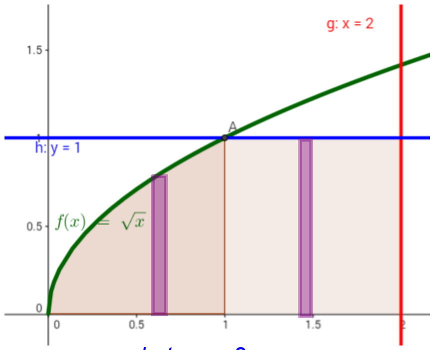
6.1 Area between Two Curves

G: $f(x) = \sqrt{x}$, $h(x) = 1$, $x=2$
x-Axis
F: area of bounded region



To use a **vertical reference** rectangle, need to break up the region into 2 parts.

$$\int_0^1 f(x)\,dx + \int_1^2 1\,dx$$



area between 2 curves

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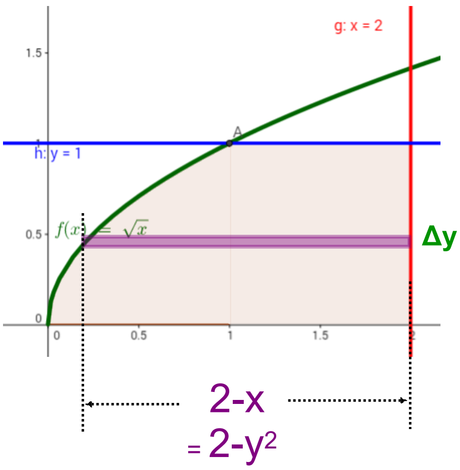
6.1 Area between Two Curves

G: $f(x) = \sqrt{x}$, $h(x) = 1$, $x=2$
x-Axis
F: area of bounded region

Use a **horizontal reference** rectangle instead. Then can do the whole region at once, using dy.

Need x from f(x):
$y = \sqrt{x}$ so $x = y^2$



$$\int_0^1 (2-y^2)\,dy = 2y - \frac{y^3}{3}\Big]_0^1$$

$$= 2 - 1/3 - 0 = 5/3$$

area between 2 curves

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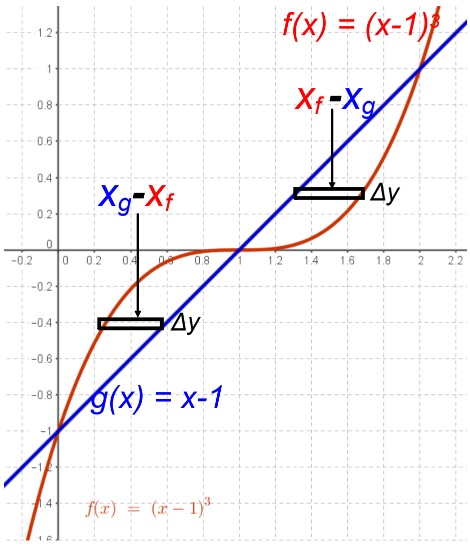
6.1 Area between Two Curves

G: $f(x) = (x-1)^3$, $g(x) = x-1$

F: area of bounded region



area between 2 curves

6.1 Area between Two Curves

G: $f(x) = (x-1)^3$, $g(x) = x-1$

F: area of bounded region

$\sqrt[3]{y} = x_f - 1$ $\quad x_g = y - 1$

$x_f = \sqrt[3]{y} + 1$

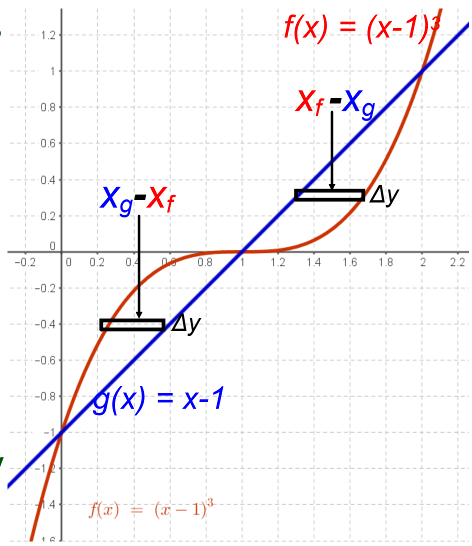
$A = \int_{-1}^{0} [(y-1)-(\sqrt[3]{y} + 1)]dy + \int_{0}^{1}[(\sqrt[3]{y} + 1)-(y-1)]dy$

$A = \int_{-1}^{0}(y - y^{1/3} - 2)dy + \int_{0}^{1}(y^{1/3} - y + 2)dy$

$= \left.\frac{y^2}{2} - \frac{y^{4/3}}{4/3}\right]_{-1}^{0} + \left.\frac{y^{4/3}}{4/3} - \frac{y^2}{2}\right]_{0}^{1}$

$= 0 - [1/2 - 3/4] + [3/4 - 1/2 - 0]$

$= -1/2 + 3/4 + 3/4 - 1/2 = 3/2 - 1 = 1/2$



6.1 Area between Two Curves

Summary of Finding Area betw 2 Curves

1. Sketch the curves and the find points of Intersection

2. Use the sketch to determine which integral to use:
   - If each curve passes the vertical line test, then Use a vertical rectangle, the x variable, and dx:

     $A = \int$ (upper - lower) dx

   - If a curve fails the vertical line test but passes the horizontal line test, then use a horizontal rectangle, the y variable, and dy:

     $A = \int$ (right - left) dy

3. If the bounded area contains more than one distinct region, write the area as the sum of the areas of each distinct region.

4. Limits of Integration:
   - Use the coordinates of the points of intersection.
   - If $x=k_1$ or $y=k_2$ is given this may be one of the limits.

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