3.3 Product and Quotient Rules of Differentiation

GOALS:

1. For functions that are not simple, recognize those with products and/or quotients.
2. Learn the Product Rule and when to use it.
3. Learn the Quotient Rule and when to use it.

Study 3.3 # 111, 117, 139, 143-147

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3.3 Product and Quotient Rules of Differentiation

$y = x^5 + x^3$ F: $\frac{dy}{dx}$

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3.3 Product and Quotient Rules of Differentiation

$y = x^5 + x^3$

$y = x^3(x^2+1)$ F: $\frac{dy}{dx}$

How do find derivative?

3.3 Product and Quotient Rules of Differentiation

$y = x^3(x^2+1)$ F: $\frac{dy}{dx}$

How do find derivative?

$(3x^2)(2x)$

NO !!

$y = x^5 + x^3$

3.3 Product and Quotient Rules of Differentiation

$y = x^3(x^2+1)$ F: $\frac{dy}{dx}$

How do find derivative?

simplify first

$y = x^5 + x^3$

$x^m x^n = x^{m+n}$

$\frac{dy}{dx} = 5x^4 + 3x^2$

$\neq 6x^3$

3.3 Product and Quotient Rules of Differentiation

$y = (x-1)(x+1)$ F: $\frac{dy}{dx}$

How do find derivative?

3.3 Product and Quotient Rules of Differentiation

$y = (x-1)(x+1)$ F: $\frac{dy}{dx}$

How do find derivative?

$y = x^2 - 1$

$\frac{dy}{dx} = 2x$

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3.3 Product and Quotient Rules of Differentiation

$y = \sqrt{x}(x+1)$ F: $\frac{dy}{dx}$

How do find derivative?

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3.3 Product and Quotient Rules of Differentiation

$y = \sqrt{x}\,(x+1)$ How do find derivative?

F: $\frac{dy}{dx}$ simplify first

$y = \sqrt{x}\,x + \sqrt{x}$

$= x^{1/2}x^1 + x^{1/2}$

$= x^{3/2} + x^{1/2}$

$\frac{dy}{dx} = \frac{3}{2}x^{1/2} + \frac{1}{2}x^{-1/2}$

$= \frac{3}{2}x^{1/2} + \frac{1}{2x^{1/2}}$

$x^m x^n = x^{m+n}$

$= \frac{3x^{1/2}}{2}\cdot\frac{x^{1/2}}{x^{1/2}} + \frac{1}{2x^{1/2}}$

$= \frac{3x+1}{2x^{1/2}} = \frac{3x+1}{2\sqrt{x}}$

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3.3 Product and Quotient Rules of Differentiation

$y = (x+1)(\sqrt{x+4})$ | $y = x(x+3)^6$

What about a product we cannot expand, or is too difficult or tedious to expand?

We need a rule to handle these.

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3.3 Product and Quotient Rules of Differentiation

Product Rule

PRODUCT RULE:

If f and g are both differentiable functions,

then $\frac{d}{dx}[f(x)\,g(x)] = f(x)\,g'(x) + g(x)\,f'(x)$

$\frac{dy}{dx} = x^3(2x) + (x^2+1)(3x^2)$
$= 2x^4 + 3x^4 + 3x^2 = 5x^4 + 3x^2$

$y = x^5 + x^3$
$\frac{dy}{dx} = 5x^4 + 3x^2$

3.3 Product and Quotient Rules of Differentiation

$y = (x-1)(x+1)$ F: $\frac{dy}{dx}$

Even though do not need the Product Rule, it should work to find the derivative.

The easy way:

$y = x^2 - 1$
$\frac{dy}{dx} = 2x$

3.3 Product and Quotient Rules of Differentiation

$y = (x-1)(x+1)$ F: $\frac{dy}{dx}$

The easy way:

$y = x^2 - 1$

$\frac{dy}{dx} = 2x$

Even though do not need the Product Rule, it should work to find the derivative.

$$\frac{dy}{dx} = (x - 1)(1) + (x + 1)(1)$$

$$\frac{dy}{dx} = x - 1 + x + 1 = 2x$$

3.3 Product and Quotient Rules of Differentiation

Product Rule

$y = x^3(x^2+1)$ F: $\frac{dy}{dx}$

$$\frac{d}{dx}[f(x)\,g(x)] = f(x)\,g'(x) + g(x)\,f'(x)$$

$\frac{dy}{dx} = x^3(2x) + (x^2+1)(3x^2)$
$= 2x^4 + 3x^4 + 3x^2 = 5x^4 + 3x^2$

3.3 Product and Quotient Rules of Differentiation

Product Rule

$y = x^3(x^2+1)$ F: $\frac{dy}{dx}$

$$\frac{d\,[\,f(x)\,g(x)\,]}{dx} = f(x)\,g'(x) + g(x)\,f'(x)$$

$$\frac{dy}{dx} = x^3(2x) + (x^2+1)(3x^2)$$

$$= 2x^4 + 3x^4 + 3x^2$$

$$= 5x^4 + 3x^2$$

$y = x^5 + x^3$

$\frac{dy}{dx} = 5x^4 + 3x^2$

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3.3 Product and Quotient Rules of Differentiation

$$\frac{d\,[\,f(x)g(x)\,]}{dx} = f(x)\,g'(x) + g(x)\,f'(x)$$

112. $f(x) = (x+2)(2x^2-3)$

Product Rule

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3.3 Product and Quotient Rules of Differentiation

$\frac{d}{dx}[f(x)g(x)] = f(x)\,g'(x) + g(x)\,f'(x)$

112. $f(x) = (x+2)(2x^2-3)$

$f'(x) = (x+2)\frac{d(2x^2-3)}{dx} + (2x^2-3)\frac{d(x+2)}{dx}$

$= (x+2)(4x) + (2x^2-3)(1)$

$= 4x^2 + 8x + 2x^2 - 3$

$= 6x^2 + 8x - 3$

expanding:
$f(x) = 2x^3 + 4X^2 - 3X - 6$

Product Rule

3.3 Product and Quotient Rules of Differentiation

$\frac{d}{dx}[f(x)g(x)] = f(x)\,g'(x) + g(x)\,f'(x)$

$f(x) = (6x+5)(x^3-2)$ F: $f'(x)$

Product Rule

3.3 Product and Quotient Rules of Differentiation

Product Rule

$$\frac{d}{dx}[f(x)g(x)] = f(x)\,g'(x) + g(x)\,f'(x)$$

$f(x) = (6x+5)(x^3-2)$ F: $f'(x)$

$$f'(x) = (6x+5)\frac{d(x^3-2)}{dx} + (x^3-2)\frac{d(6x+5)}{dx}$$

$$= (6x+5)(3x^2) + (x^3-2)(6)$$

$$= 18x^2 + 15x^2 + 6x^3 - 12$$

$$= 24x^3 + 15x^2 - 12$$

3.3 Product and Quotient Rules of Differentiation

$$\frac{d}{dx}[f(x)g(x)] = f(x)\,g'(x) + g(x)\,f'(x)$$

$g(s) = \sqrt{s}\,(4-s^2)$ F: $g'(s)$

Product Rule

3.3 Product and Quotient Rules of Differentiation

Product Rule

$\frac{d}{dx}[f(x)g(x)] = f(x)\,g'(x) + g(x)\,f'(x)$

$y = g(s) = \sqrt{s}\,(4 - s^2)$ F: $g'(s)$

$= s^{1/2}(4 - s^2)$

$g'(s) = s^{1/2}\frac{d(4-s^2)}{ds} + (4-s^2)\frac{d(s^{1/2})}{ds}$

$= s^{1/2}(-2s) + (4-s^2)\left(\frac{1}{2}s^{-1/2}\right)$

$= -2s^{3/2} + (4-s^2)\left(\frac{1}{2s^{1/2}}\right)$

$= -2s^{3/2} + \frac{4-s^2}{2s^{1/2}}$

$= \frac{-4s^2}{2s^{(1/2)}} + \frac{4-s^2}{2s^{(1/2)}} = \frac{4-5s^2}{2s^{(1/2)}}$

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3.3 Product and Quotient Rules of Differentiation

Product Rule

$\frac{d}{dx}[f(x)g(x)] = f(x)\,g'(x) + g(x)\,f'(x)$

EXTRA

G: $y = 2x(4 - x^3)$ F: $\frac{dy}{dx}$

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3.3 Product and Quotient Rules of Differentiation

Product Rule

$$\frac{d}{dx}[f(x)g(x)] = f(x)\,g'(x) + g(x)\,f'(x)$$

G: $y = 2x(4-x^3)$ F: $\frac{dy}{dx}$

$$\frac{dy}{dx} = 2x(-3x^2) + (4-x^3)(2)$$

$$= -6x^3 + 8 - 2x^3$$

$$= -8x^3 + 8$$

$y = 4x - 2x^4$

$\frac{dy}{dx} = 4 - 8x^3$

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3.3 Product and Quotient Rules of Differentiation

Quotient Rule

$g(t) = \frac{t^2+2}{2t-7}$ $= (t^2+2)(2t-7)^{-1}$

QUOTIENT RULE:

$$\frac{d[N/D]}{dx} = \frac{D\,\frac{d(N)}{dx} - N\,\frac{d(D)}{dx}}{D^2}$$

If f and g are both differentiable functions, then the quotient $\frac{f}{g}$ is differentiable at all x such that $g(x) \neq 0$ and

$$\frac{d\left[\frac{f(x)}{g(x)}\right]}{dx} = \frac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2}$$

$\frac{DdN - NdD}{D^2}$

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3.3 Product and Quotient Rules of Differentiation

Quotient Rule

QUOTIENT RULE:

If f and g are both differentiable functions,

then the quotient $\frac{f}{g}$ is differentiable at all x

such that $g(x) \neq 0$ and

$$\frac{d\left[\frac{f(x)}{g(x)}\right]}{dx} = \frac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2}$$

$$\frac{d\,[\,N / D\,]}{dx} = \frac{D\,\frac{d(N)}{dx} - N\,\frac{d(D)}{dx}}{D^2}$$

$$= \frac{D\,d(N) - N\,d(D)}{D^2}$$

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$f(x) = \frac{x}{x+1}$

$$\frac{d\left[\frac{f(x)}{g(x)}\right]}{dx} = \frac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2}$$

$$= \frac{D\,dN - N\,dD}{D^2}$$

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$$f(x) = \frac{x}{x+1}$$

$$\frac{d\left[\frac{f(x)}{g(x)}\right]}{dx} = \frac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2}$$

$$= \frac{D\,dN - N\,dD}{D^2}$$

$$f'(x) = \frac{(x+1)\frac{dx}{dx} - x\frac{d(x+1)}{dx}}{(x+1)^2}$$

$$f'(x) = \frac{(x+1)(1) - x(1)}{(x+1)^2}$$

$$= \frac{x+1-x}{(x+1)^2} = \frac{1}{(x+1)^2}$$

using definition of derivative: shown to appreciate the rule as being less complex than the definition to find the derivative

$$f(x) = \frac{x}{x+1}$$

$$f'(x) = \lim_{h\to 0} \frac{\frac{x+h}{x+h+1} - \frac{x}{x+1}}{h} \qquad \frac{(x+h+1)(x+1)}{(x+h+1)(x+1)}$$

$$\frac{(x+h)(x+1) - x(x+h+1)}{h(x+h+1)(x+1)}$$

$$\frac{x^2+hx+x+h-x^2-xh-x}{h(x+h+1)(x+1)}$$

$$\frac{h}{h(x+h+1)(x+1)}$$

$$\therefore \lim_{h\to 0} \frac{1}{(x+h+1)(x+1)} = \frac{1}{(x+1)^2}$$

3.3 Product and Quotient Rules of Differentiation

Quotient Rule

$$g(t) = \frac{t^2+2}{2t-7}$$

$$\frac{d\left[\frac{f(x)}{g(x)}\right]}{dx} = \frac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2}$$

$$= \frac{D\,dN - N\,dD}{D^2}$$

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3.3 Product and Quotient Rules of Differentiation

Quotient Rule

$$h(t) = \frac{t^2+2}{2t-7}$$

$$\frac{d\left[\frac{f(x)}{g(x)}\right]}{dx} = \frac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2}$$

$$= \frac{D\,dN - N\,dD}{D^2}$$

$$h'(t) = \frac{(2t-7)\frac{d}{dt}(t^2+2) - (t^2+2)\frac{d}{dt}(2t-7)}{(2t-7)^2}$$

$$= \frac{(2t-7)(2t) - (t^2+2)(2)}{(2t-7)^2}$$

$$= \frac{4t^2-14t-2t^2-4}{(2t-7)^2}$$

$$= \frac{2t^2-14t-4}{(2t-7)^2}$$

no common factors;
cannot reduce to lowest terms
(simplify)

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$f(x) = \dfrac{x^2+2}{2x-7}$

using definition of derivative: shown to appreciate the rule as being less complex than the definition to find the derivative

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$f(x+h) = \dfrac{(x+h)^2+2}{2(x+h)-7}$

$$\frac{\dfrac{(x+h)^2+2}{2(x+h)-7} - \dfrac{x^2+2}{2x-7}}{h} \cdot \frac{[2(x+h)-7][2x-7]}{[2(x+h)-7][2x-7]}$$

$$= \frac{[(x+h)^2+2][2x-7] - (x^2+2)[2(x+h)-7]}{h[2(x+h)-7][2x-7]}$$

$$= \frac{(x+h)^2(2x-7) + 4x - 14 - [(x^2+2)(2x+2h-7)]}{h[2(x+h)-7][2x-7]}$$

$$= \frac{(x^2+2xh+h^2)(2x-7) + 4x - 14 - [2x^3+2hx^2-7x^2+4x+4h-14]}{h[2(x+h)-7][2x-7]}$$

$$= \frac{2x^3+4hx^2+2h^2x-7x^2-14hx-7h^2+4x-14}{h[2(x+h)-7][2x-7]}$$

$$\frac{-2x^3-2hx^2+7x^2-4x+14-4h}{h[2(x+h)-7][2x-7]}$$

$$\frac{2hx^2+2h^2x-14hx-7h^2-4h}{h[2(x+h)-7][2x-7]} = \frac{2x^2+2hx-14x-7h-4}{2(x+h)-7][2x-7]}$$

$$\therefore f'(x) = \lim_{h\to 0} \frac{2x^2+2hx-14x-7h-4}{2(x+h)-7][2x-7]} = \frac{2x^2-14x-4}{(2x-7)^2}$$

3.3 Product and Quotient Rules of Differentiation

Quotient Rule

$$\frac{d\left[\frac{f(x)}{g(x)}\right]}{dx} = \frac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2}$$

$$= \frac{D\,dN - N\,dD}{D^2}$$

116. $f(x) = \dfrac{x^2+4}{x^2-4}$

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3.3 Product and Quotient Rules of Differentiation

Quotient Rule

116. $f(x) = \dfrac{x^2+4}{x^2-4}$

$$\frac{d\left[\frac{f(x)}{g(x)}\right]}{dx} = \frac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2}$$

= D dN - N dD / D²

$$f'(x) = \frac{(x^2-4)\frac{d}{dx}(x^2+4) - (x^2+4)\frac{d}{dx}(x^2-4)}{(x^2-4)^2}$$

$$= \frac{(x^2-4)(2x) - (x^2+4)(2x)}{(x^2-4)^2}$$

$$= \frac{2x^3 - 8x - 2x^3 - 8x}{(x^2-4)^2} = \frac{-16x}{(x^2-4)^2}$$

3.3 Product and Quotient Rules of Differentiation

Quotient Rule

$G: y(x) = \dfrac{2x+5}{\sqrt{x}}$ $x > 0$ $F: f'(x)$

Can use:

1. Simplification
2. Product Rule
3. Quotient Rule

3.3 Product and Quotient Rules of Differentiation

Quotient Rule

Quotient Rule

29. $G: f(x) = \frac{2x+5}{\sqrt{x}}$ $F: f'(x)$

$f(x) = x^{-1/2}(2x+5) = 2x^{1/2} + 5x^{-1/2}$

$f'(x) = 2\cdot\frac{1}{2}x^{-1/2} + 5\left(-\frac{1}{2}x^{-3/2}\right) = x^{-1/2} - \frac{5}{2}x^{-3/2}$

$= \frac{1}{\sqrt{x}} - \frac{5}{2x\sqrt{x}} = \frac{2x-5}{2x\sqrt{x}}$

$f(x) = (2x+5)x^{-1/2}$

$f'(x) = (2x+5)\left(-\frac{1}{2}x^{-3/2}\right) + \left(x^{-1/2}\right)(2)$

$= -\frac{(2x+5)}{2x^{3/2}} + \frac{2}{x^{1/2}}\cdot\frac{2x}{2x} = \frac{-2x-5+4x}{2x\sqrt{x}} = \frac{2x-5}{2x\sqrt{x}}$

$f(x) = \frac{2x+5}{\sqrt{x}} = \frac{2x+5}{x^{1/2}}$

$= \frac{D\,dN - N\,dD}{D^2}$

$f'(x) = \frac{x^{1/2}(2) - (2x+5)\left(\frac{1}{2}x^{-1/2}\right)}{\left(x^{1/2}\right)^2}$

$= \frac{\left(2x^{1/2} - \frac{2x+5}{2x^{1/2}}\right)}{x}\cdot\frac{2x^{1/2}}{2x^{1/2}} = \frac{4x-(2x+5)}{2x^{3/2}}$

$= \frac{2x-5}{2x^{3/2}}$

3.3 Product and Quotient Rules of Differentiation

derivative practice

$y = x^4 - 3x^2 + 1$ $F: y''$ $\frac{d^2y}{dx^2} = f''(x)$

3.3 Product and Quotient Rules of Differentiation

derivative practice

$y = x^4 - 3x^2 + 1$ F: y'' $\frac{d^2y}{dx^2} = f''(x)$

$\frac{dy}{dx} = 4x^3 - 6x$

$\frac{d^2y}{dx^2} = 12x^2 - 6$

$\frac{d^3y}{dx^3} = 24x$

$\frac{d^4y}{dx^4} = 24$ $\frac{d^5y}{dx^5} = 0$

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3.3 Product and Quotient Rules of Differentiation

derivative practice

$h(t) = \sqrt{t}(1 - t^2)$ F: $h'(t)$

$= t^{\frac{1}{2}}(1 - t^2) = t^{\frac{1}{2}} - t^{\frac{5}{2}}$

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3.3 Product and Quotient Rules of Differentiation

derivative practice

$h(t) = \sqrt{t}(1-t^2)$

$h(t) = t^{1/2}(1-t^2)$

$h'(t) = t^{1/2}\frac{d(1-t^2)}{dt} + (1-t^2)\frac{d(t^{1/2})}{dt}$

$= t^{1/2}(-2t^1) + (1-t^2)\left(\frac{1}{2}t^{-1/2}\right)$

$\frac{2t^{1/2}(-2t^{3/2})}{2t^{1/2}} + \frac{1-t^2}{2t^{1/2}} = \frac{-4t^2+1-t^2}{2t^{1/2}} = \frac{-5t^2+1}{2t^{1/2}}$

$= t^{1/2}(1-t^2) = t^{1/2} - t^{5/2}$

$h'(t) = \frac{1}{2}t^{-1/2} - \frac{5}{2}t^{3/2}$

$= \frac{1}{2t^{1/2}} - \frac{5}{2}t^{3/2}\cdot\frac{t^{1/2}}{t^{1/2}}$

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3.3 Product and Quotient Rules of Differentiation

derivative practice

G: $f(x) = 8x^6 - 10x^5 + 5x^3$ F: $f''(x)$

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3.3 Product and Quotient Rules of Differentiation

G: $f(x) = 8x^6 - 10x^5 + 5x^3$ F: $f''(x)$

$f'(x) = 48x^5 - 50x^4 + 15x^2$

$f''(x) = 240x^4 - 200x^3 + 30x$

3.3 Product and Quotient Rules of Differentiation

derivative practice

Quotient Rule

G: $f(x) = x\left(1 - \frac{4}{x+3}\right)$ F: $f'(x)$

3.3 Product and Quotient Rules of Differentiation

Quotient Rule

G: $f(x) = x\left(1 - \frac{4}{x+3}\right)$ F: $f'(x)$

$f(x) = x - \frac{4x}{x+3} = \frac{x(x+3) - 4x}{x+3}$

$f'(x) = 1 - \left[\frac{(x+3)(4) - 4x(1)}{(x+3)^2}\right] = 1 - \left[\frac{4x+12-4x}{(x+3)^2}\right]$

$= 1 - \frac{12}{(x+3)^2} = \frac{(x+3)^2 - 12}{(x+3)^2}$

To reduce to lower terms (RLT), need a factor of (x+3) in numerator which would cause numerator to =0 when x=-3. But when x=-3, num = -12. So, no common factor, no RLT.

3.3 Product and Quotient Rules of Differentiation

derivative practice

G: $f(x) = x + 32x^{-2}$ F: $f''(x)$

3.3 Product and Quotient Rules of Differentiation

G: $f(x) = x + 32x^{-2}$ F: $f''(x)$

$f'(x) = 1 + 32(-2x^{-3})$ $f'(x)$

$= 1 - 64x^{-3}$

$f''(x) = (-64)(-3x^{-4})$ $f''(x)$

$= 192x^{-4}$

$f''' = -768x^{-5}$ $f'''(x)$

$y = x^4 + 3x^2 - 1$ F: $\frac{d^2y}{dx^2}$

$\frac{dy}{dx} = 4x^3 + 6x$

$\frac{d^2y}{dx^2} = 12x^2 + 6$

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