2.1 Tangent, Velocity, Area

GOALS: Understand
1. What are tangent lines?
2. What are secant lines?
3. How do slopes of tangent lines relate to slopes of secant lines?
4. How does the slope of a tangent line relate to the slope of the function at the point in common to both.
5. What is average velocity?
6. What is instantaneous velocity?
7. How can we find areas beneath curves?

Study 2.1 # 1, 2, 3, 7, 9, 16, 17, 24, 25

classnotes

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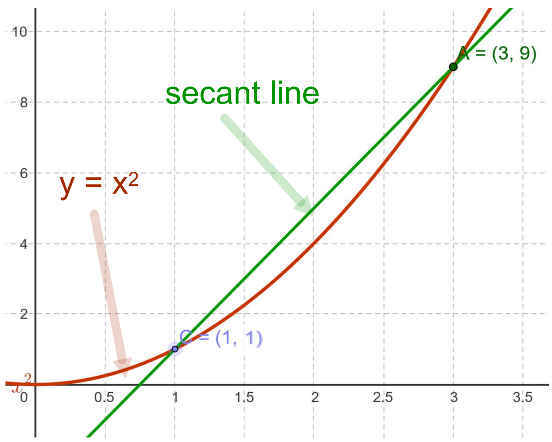
2.1 Tangent

What is a secant line ?

Secant Line:

■A line that intersects a curve in two or more points.



tangent line

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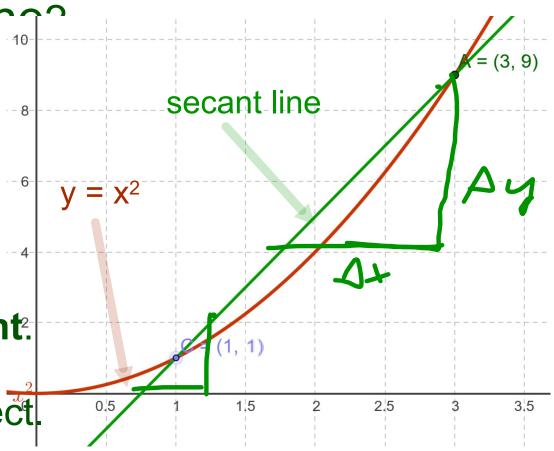
2.1 Tangent

$y = mx + b$

hat is the slope of a secant line?

$$m_{sec} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{9 - 1}{3 - 1} = \frac{8}{2} = 4$$



Slope is the rate of change of y with respect to x.
For a **straight line**, this is **constant**.
ie: y changes at the same rate no matter what values of x we select.

For a **curve**, this is **variable**.
ie: we expect y to change at a different rate for different values of x

Calculus:
How do we find the rate of change of y with respect to x for curves?

tangent line

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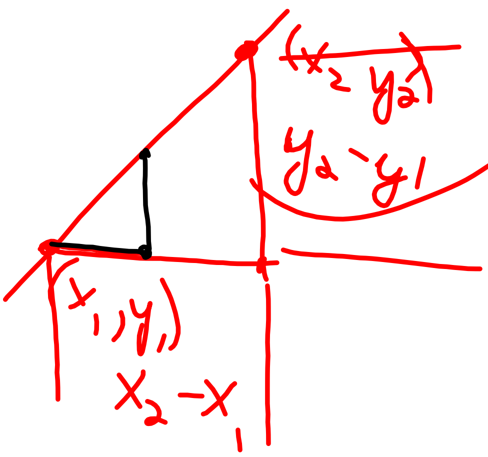
$$m = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$$



$$m = \frac{\Delta s}{\Delta t} = \frac{s_2 - s_1}{t_2 - t_1}$$

2.1 Tangent

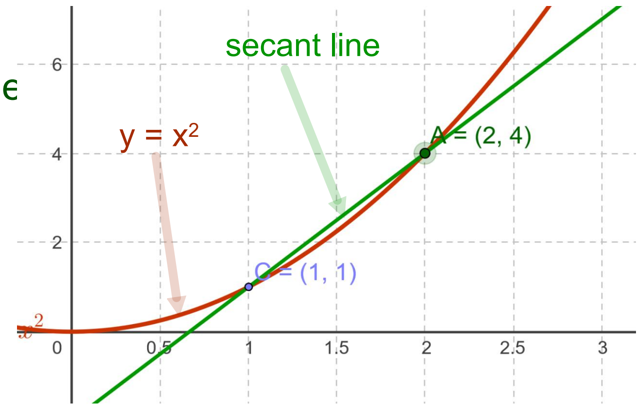
Calculus:
Examine the slope of the secant line as point A approaches point C.

What is the slope of a secant line?

- $m_{sec} = \dfrac{y_2 - y_1}{x_2 - x_1} = \dfrac{9 - 1}{3 - 1} = \dfrac{8}{2} = 4$

What is the slope of a secant line

- $m_{sec} = \dfrac{y_2 - y_1}{x_2 - x_1} = \dfrac{4 - 1}{2 - 1} = \dfrac{3}{1} = 3$



tangent line

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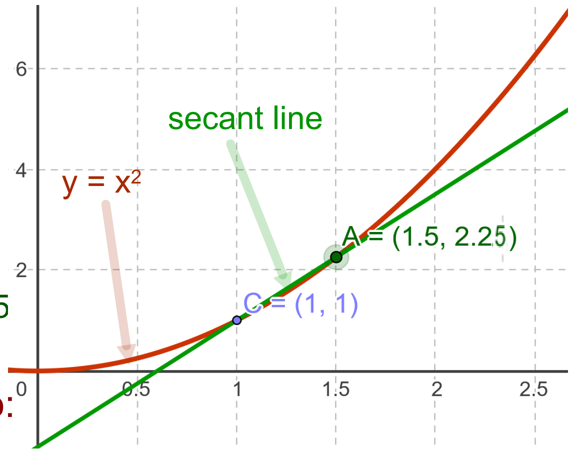
2.1 Tangent

What is the slope of a secant line?

- $m_{sec} = \dfrac{y_2 - y_1}{x_2 - x_1} = \dfrac{9 - 1}{3 - 1} = \dfrac{8}{2} = 4$
- $m_{sec} = \dfrac{y_2 - y_1}{x_2 - x_1} = \dfrac{4 - 1}{2 - 1} = \dfrac{3}{1} = 3$

What is the slope of a secant line?

- $m_{sec} = \dfrac{y_2 - y_1}{x_2 - x_1} = \dfrac{2.25 - 1}{1.5 - 1} = \dfrac{1.25}{0.5} = 2.5$



slope of a secant line from C(1,1) to:

A(x,y)	m_{sec}
(3,9)	4
(2,4)	3
(1.5,2.25)	2.5
(1.1,1.21)	2.1
(1.01,1.0201)	2.01

$\dfrac{1.0201 - 1}{1.01 - 1}$

$\dfrac{0.0201}{0.01} = 2.01$

$\dfrac{1.21 - 1}{1.1 - 1}$

$\dfrac{0.21}{0.1}$

What value does m_{sec} approach as A(x,y) gets closer to C(1,1)?

tangent line

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2.1 Tangent

What is the slope of a secant line?

- $m_{sec} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{9 - 1}{3 - 1} = \frac{8}{2} = 4$
- $m_{sec} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{4 - 1}{2 - 1} = \frac{3}{1} = 3$

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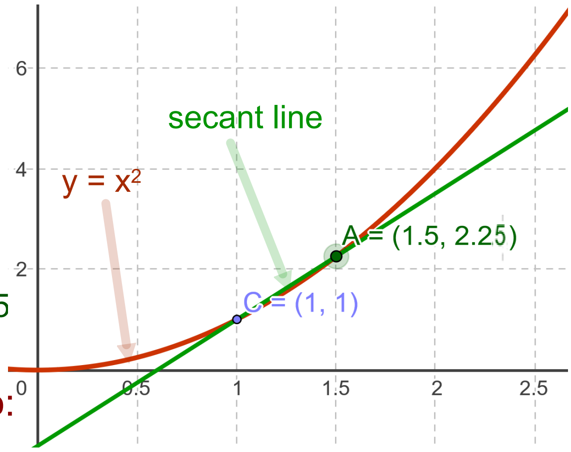

slope of a secant line from C(1,1) to:

A(x,y)	m_{sec}
(3,9)	4
(2,4)	3
(1.5,2.25)	2.5
(1.1,1.21)	2.1
(1.01,1.02)	2.01

What value does m_{sec} approach as A(x,y) gets closer to C(1,1)?

$m_{sec} \to 2$

$m_{sec} \to m_{tan}$

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tangent line

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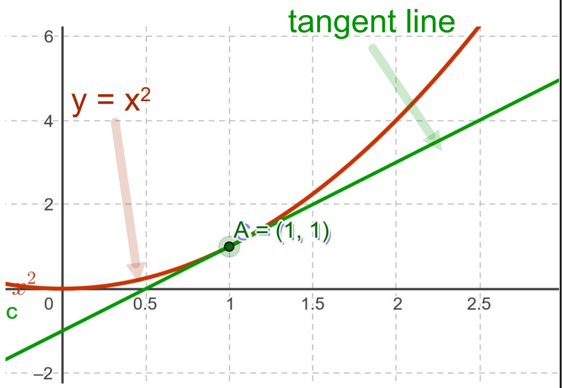
2.1 Tangent

What is the slope of a secant line?

- $m_{sec} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{9 - 1}{3 - 1} = \frac{8}{2} = 4$
- $m_{sec} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{4 - 1}{2 - 1} = \frac{3}{1} = 3$
- $m_{sec} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{2.25 - 1}{1.5 - 1} = \frac{1.25}{0.5} = 2.5$



slope of a secant line from C(1,1) to:

A(x,y)	m_{sec}
(3,9)	4
(2,4)	3
(1.5,2.25)	2.5
(1.1,1.21)	2.1
(1.01,1.02)	2.01
(1.0001,1.00020001)	? 2.0001
(1,1)	?

m_{tan}

What value does m_{sec} approach as A(x,y) gets closer to C(1,1)?

$m_{sec} \to 2$

$$\frac{0.00020001}{0.0001} = 2.0001$$

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tangent line

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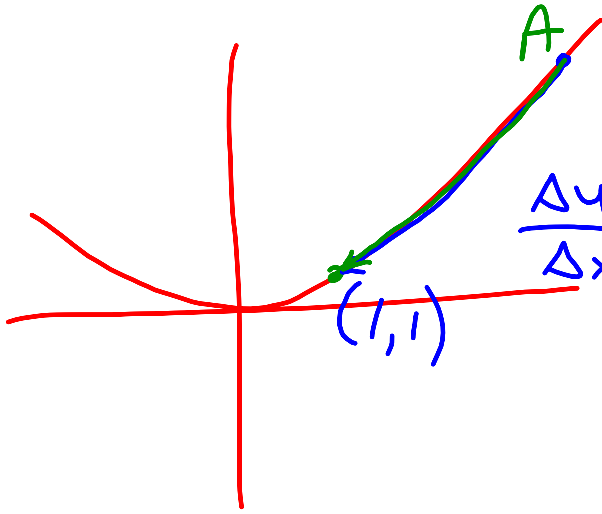


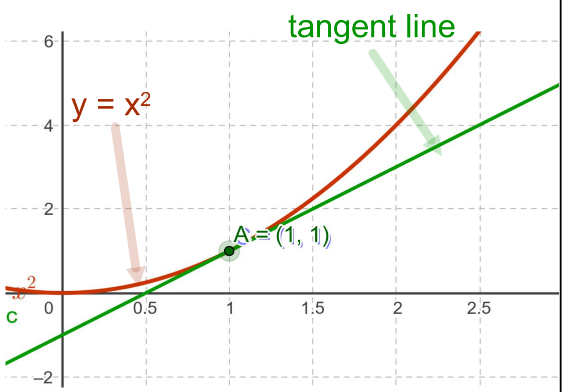
$$\frac{\Delta y}{\Delta x} = \frac{y_2 - 1}{x_2 - 1} = \frac{1-1}{1-1}$$

A(1,1)

2.1 Tangent

What is the slope of a secant line?

- $m_{sec} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{9-1}{3-1} = \frac{8}{2} = 4$
- $m_{sec} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{4-1}{2-1} = \frac{3}{1} = 3$
- $m_{sec} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{2.25-1}{1.5-1} = \frac{1.25}{0.5} = 2.5$



slope of a secant line from C(1,1) to:

A(x,y)	m_{sec}
(3,9)	4
(2,4)	3
(1.5,2.25)	2.5
(1.1,1.21)	2.1
(1.01,1.02)	2.01
(1.0001,1.00020001)	2.0001
(1,1)	?

What value does m_{sec} approach as A(x,y) gets closer to C(1,1)?

$m_{sec} \to 2$

But $\frac{1-1}{1-1}$ DNE DIV BY 0

$m_{sec} \to m_{tan}$

but cannot use slope formula to find

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tangent line

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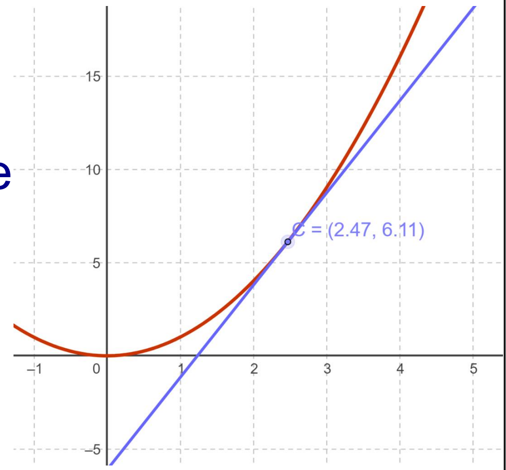
2.1 Tangent

What is a tangent line ?

Tangent Line:

- A line that touches a curve at a point without crossing over.
- A line which intersects a (differentiable) curve at a point where the slope of the curve equals the slope of the line.



tangent line

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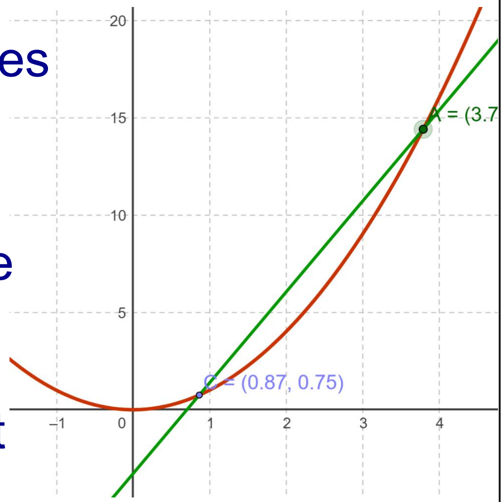
2.1 Tangent

How do slopes of tangent lines relate to slopes of secant lines?

- They are only related if one of the points defining the secant line is also a point on the curve and on the tangent line.



- As the point not in common approaches the common point, the slope of the secant approaches the slope of the tangent.

tangent line

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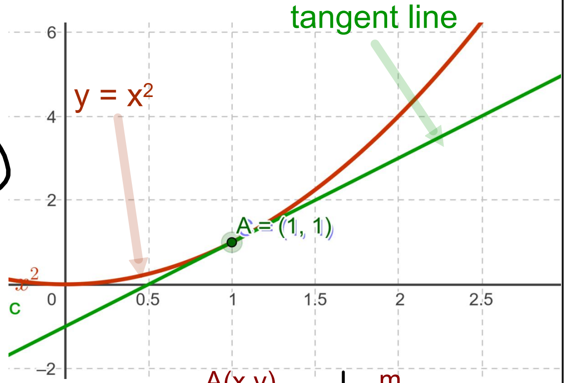
2.1 Tangent

m_{sec} --> 2

conclude $m_{tan} = 2$

Find: equation of tangent line to $y = x^2$ at C(1,1)?



A(x,y)	m_{sec}
(3,9)	4
(2,4)	3
(1.5,2.25)	2.5
(1.1,1.21)	2.1
(1.01,1.02)	2.01
(1.0001,1.00020001)	2.0001
(1,1)	? DNE

$y = m_{tan} x + b$

$y - y_1 = m_{tan}(x - x_1)$

$y - 1 = 2(x - 1)$

$y - 1 = 2x - 2$

$y = 2x - 1$

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tangent line

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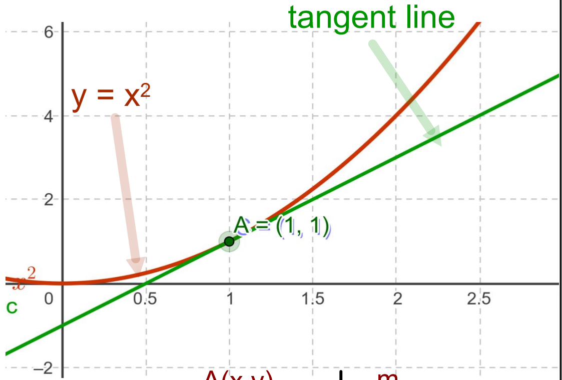
2.1 Tangent

m_{sec} --> 2

conclude $m_{tan} = 2$

Find: equation of tangent line to $y = x^2$ at C(1,1)?



A(x,y)	m_{sec}
(3,9)	4
(2,4)	3
(1.5,2.25)	2.5
(1.1,1.21)	2.1
(1.01,1.02)	2.01
(1.0001,1.00020001)	2.0001
(1,1)	? DNE

Point-slope form of straight line:

$y - y_1 = m_{tan}(x - x_1)$

$y - 1 = 2(x - 1)$

$y - 1 = 2x - 2$

$y = 2x - 1$ equation of tangent line to $y = x^2$ at C(1,1)

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tangent line

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2.1 Tangent

skip

$m = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{y-1}{x-1} = \frac{L2-1}{L1-1}$

(x_1, y_1) (x_2, y_2)

or the following exercises, points $P(1, 1)$ and $Q(x, y)$

e on the graph of the function $f(x) = x^3$.

$y = x^3$

$y|_{x=1} = 1$

$P(1,1)$

. **[T]** Complete the following table with the appropriate alues: y-coordinate of Q, the point $Q(x, y)$, and the slope f the secant line passing through points P and Q. Round our answer to eight significant digits.

x	y	Q(x, y)	m_{sec}
		(1.1, 1.331)	$\frac{0.331}{0.1}$

L1 (x)	L2 (y)	L3	L4
1.1	1.331	3.31	------
1.01	1.0303	3.0301	
1.001	1.003	3.003	
1.0001	1.0003	3.0003	
------	------	------	

tangent line

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tangent line.

2.1 Tangent

For the following exercises, points $P(1, 1)$ and $Q(x, y)$ are on the graph of the function $f(x) = x^3$.

4. **[T]** Complete the following table with the appropriate values: y-coordinate of Q, the point $Q(x, y)$, and the slope of the secant line passing through points P and Q. Round your answer to eight significant digits.

x	y	Q(x, y)	m_{sec}
1.1	a.	e.	i.
1.01	b.	f.	j.
1.001	c.	g.	k.
1.0001	d.	h.	l.

$m_{PQ} = \frac{y-1}{x-1} = (L2-1)/(L1-1)$

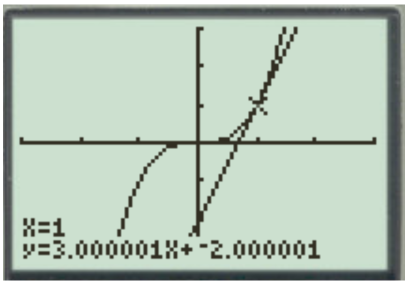
L1	L2	
1.1	1.331	------
1.01	1.0303	
1.001	1.003	
1.0001	1.0003	
------	------	

L3 =

L1 (x)	L2 (y)	L3 (m_{sec})
1.1	1.331	3.31
1.01	1.0303	3.0301
1.001	1.003	3.003
1.0001	1.0003	3.0003
------	------	

L3(5) =

$m_{tan} = 3$



As Q(x,y) gets closer and closer to P(1,1), m_{sec} approaches $m_{tan} = 3$

tangent line

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2.1 Tangent

skip

Example: 2.1 # 2 (stewart)

Cardiac monitor: number of heart eats after t minutes

t(min)	36	38	40	42	44
beats	2530	2661	2806	2948	3080

Estimate heart rate after 42 minutes using secant line between

a) t=36 and t=42 b) t=38 and t=42

c) t=40 and t=42 d) t=42 and t=44.

What are your conclusions?

tangent line

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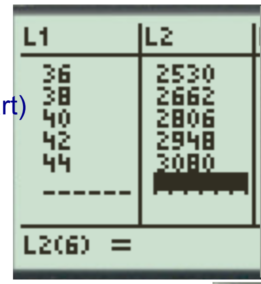
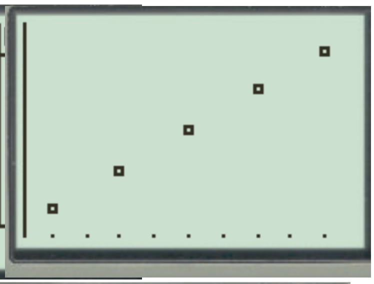
2.1 Tangent

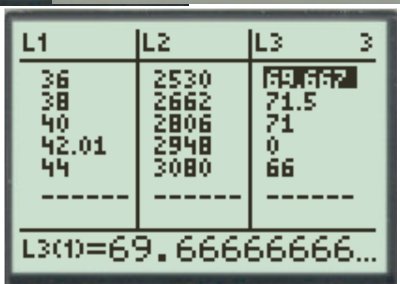
Example: 2.1 # 2 (stewart)



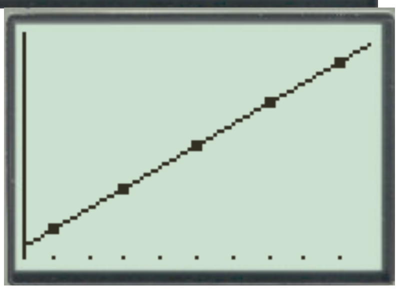


$$\frac{\Delta y}{\Delta x} = \frac{L_2 - 2948}{(L_1 - 42)}$$

but need to avoid division by zero, so change 42 in L1 to 42.01



Not really conclusive, but can average all, w/o 0; or could average the 2 closest



Get 68.5 for 2 closest.

Get 69.6 for all

Linear regression gets slope of 69.2

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2.1 Velocity

Montauk Point is 150 miles away. I am planning to go birding there this weekend. If it takes me 3 hours to get there, how fast will I travel?

What is this $\boxed{\text{speed}}$ of travel called? average $= \frac{150 \text{ mi}}{3 \text{ hr.}}$
$= 50$mph

Do I travel at that speed when I leave my driveway? When I am on the LI expressway?

How fast could I be travelling at any time during the trip? What is this called?

tangent line

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2.1 Velocity

Montauk Point is 150 miles away. I am planning to go birding there this weekend. If it takes me 3 hours to get there, how fast will I travel? $\frac{150}{3} = 50$ mph

What is this speed of travel called? average velocity, or speed (no direction)

Do I travel at that speed when I leave my driveway? When I am on the LI expressway?

No. probably drive at -5 mph backing up
No. probably drive at 55 mph or 60 mph

How fast could I be travelling at any time during the trip? What is this called? instantaneous velocity: rate of change at a specific time, t

tangent line

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2.1 Velocity

Average Velocity

s(t) is the position of an object moving along a coordinate axis at time t. The average velocity of the object over a time interval [a, t] is

$$v_{ave} = \frac{s(t)-s(a)}{t-a} = \frac{\Delta s}{\Delta t}$$

Instantaneous Velocity

The instantaneous velocity at the time $t=a$ is the value that the average velocities approach as t approaches a, assuming that the value exists.

tangent line

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2.1 Velocity

$h(1) = 15 - 4.9 = 10.1$ m

$h(t) = 15t - 4.9t^2$

$t(15 - 4.9t)$ t

For the following exercises, consider a stone tossed into the air from ground level with an initial velocity of 15 m/sec. Its height in meters at time t seconds is $h(t) = 15t - 4.9t^2$.

$(1, 10.1)$

$4.9t = 15$

$t = \frac{15}{4.9}$

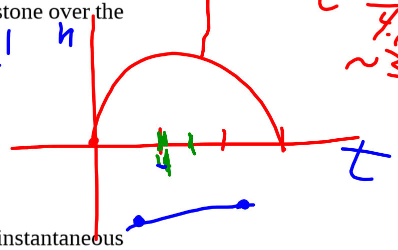
~ 3

18. **[T]** Compute the average velocity of the stone over the given time intervals.

$\frac{\Delta h}{\Delta t} = \frac{-10.1}{-1}$

a. [1, 1.05]
b. [1, 1.01]
c. [1, 1.005]
d. [1, 1.001]

10, 1051951



19. Use the preceding exercise to guess the instantaneous velocity of the stone at $t = 1$ sec.

$$= \frac{h(t) - 10.1}{t - 1}$$

$$\frac{\Delta h}{\Delta t} = \frac{h(t) - h(1)}{t - 1}$$

$$= \frac{L2 - 10.1}{L1 - 1}$$

tangent line

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$h(t) = 15t - 4.9t^2$

t	$h(t)$	$\dfrac{h(t)-h(\cancel{1})^{10.1}}{t-1}$	
1.05	10.348	$(10.348-10.1)/(1.05-1) = \dfrac{0.248}{0.05}$	4.96
1.01	10.152	$(10.152-10.1)/(1.01-1) = \dfrac{0.052}{0.01}$	5.2
1.005	10.126	$(10.126-10.1)/(1.005-1) = \dfrac{0.026}{0.005}$	5.2
1.001	10.1052	$(10.105-10.1)/(1.001-1) = \dfrac{0.0052}{0.001}$	52

average speed between t=1 and t=1.05 sec = 4.96

average speed between t=1 and t=1.01 sec = 5.2

average speed between t=1 and t=1.005 sec = 5.2

average speed between t=1 and t=1.001 sec = 5.2

Doing this by hand, we **used too few decimal places to actually get a meaningful difference among the results** - eg: three of four above are 5.2. Need to use an extra decimal place. See next page.

$h(t) = 15t - 4.9t^2$

t	$h(t)$	$\dfrac{h(t)-h(\cancel{1})^{10.1}}{t-1}$		
1.05	10.348	$(10.3478-10.1)/(1.05-1) = \dfrac{0.2478}{0.05}$	4.96	4.956
1.01	10.152	$(10.1512-10.1)/(1.01-1) = \dfrac{0.0512}{0.01}$	5.2	5.12
1.005	10.126	$(10.12659-10.1)/(1.005-1) = \dfrac{0.02659}{0.005}$	5.2	5.18
1.001	10.1052	$(10.1052-10.1)/(1.001-1) = \dfrac{0.0052}{0.001}$	52	5.20

average speed between t=1 and t=1.05 sec = 4.956

average speed between t=1 and t=1.01 sec = 5.12

average speed between t=1 and t=1.005 sec = 5.18

average speed between t=1 and t=1.001 sec = 5.20

Adding one more decimal place to the h values, we get results that discriminate between outcomes.

We see that, as t approaches the value of 1, the average speed approaching 5.2.

2.1 Velocity

For the following exercises, consider a stone tossed into the air from ground level with an initial velocity of 15 m/sec. Its height in meters at time t seconds is $h(t) = 15t - 4.9t^2$.

18. **[T]** Compute the average velocity of the stone over the given time intervals.
    a. [1, 1.05]
    b. [1, 1.01]
    c. [1, 1.005]
    d. [1, 1.001]

19. Use the preceding exercise to guess the instantaneous velocity of the stone at $t = 1$ sec.

$$\bar{v} = \frac{\Delta s}{\Delta t} = \frac{s(t) - s(1)}{t-1}$$

$$= \frac{15t - 4.9t^2 - (15 - 4.9)}{t-1}$$

$$= \frac{15t - 4.9t^2 - (10.1)}{t-1}$$

tangent line

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2.1 Velocity

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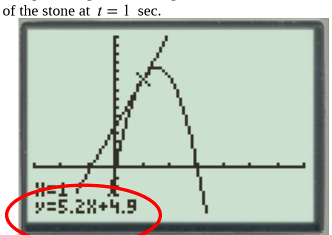
$$\bar{v} = \frac{\Delta s}{\Delta t} = \frac{s(t) - s(1)}{t-1}$$

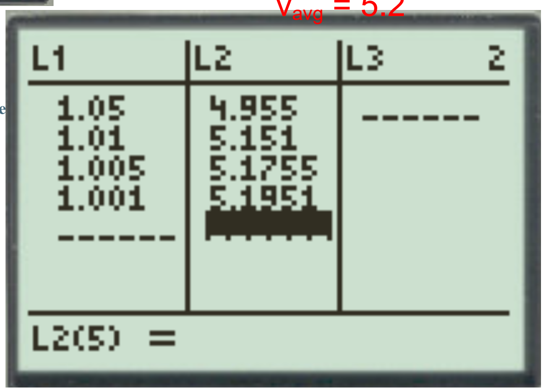
$$= \frac{15t - 4.9t^2 - (15 - 4.9)}{t-1}$$

$$= \frac{15t - 4.9t^2 - (10.1)}{t-1}$$



$$\frac{(15\,L1 - 4.9\,L1^2 - 10.1)}{(L1 - 1)}$$

$v_{avg} = 5.2$



L1	L2	L3 2
1.05	4.955	------
1.01	5.151	
1.005	5.1755	
1.001	5.1951	

L2(5) =

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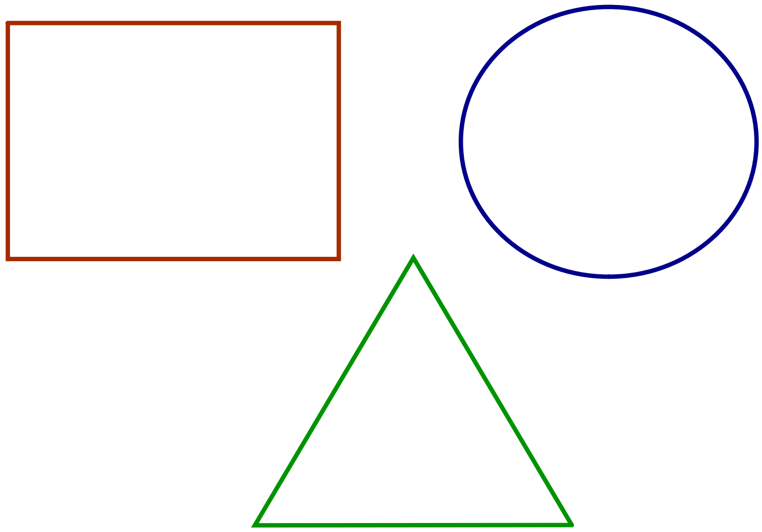
2.1 Area

How do we find the area of:



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2.1 Area

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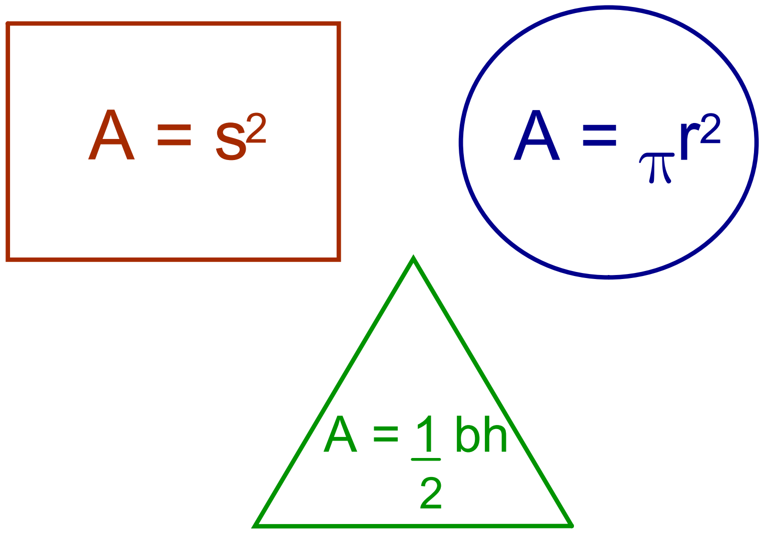

$A = s^2$

$A = \pi r^2$

$A = \frac{1}{2} bh$

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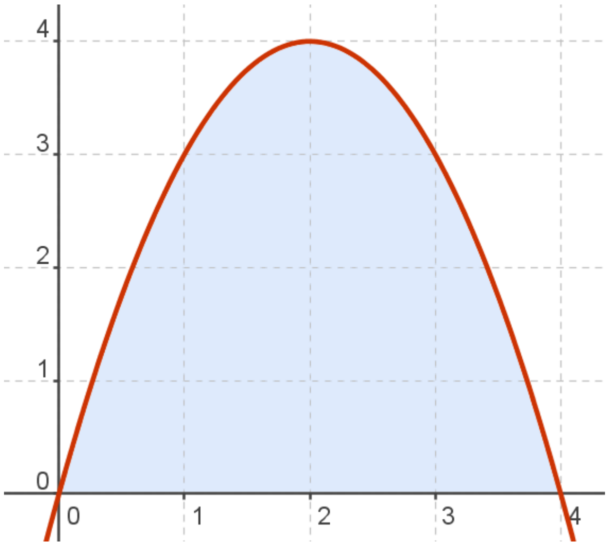
2.1 Area

What about the shaded area below?



Can we find the area with a formula or other tools from algebra or geometry?

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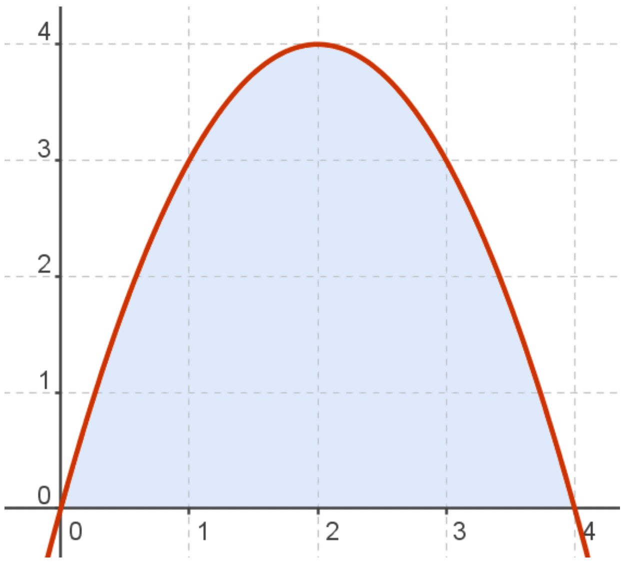
2.1 Area

How can we estimate the area?



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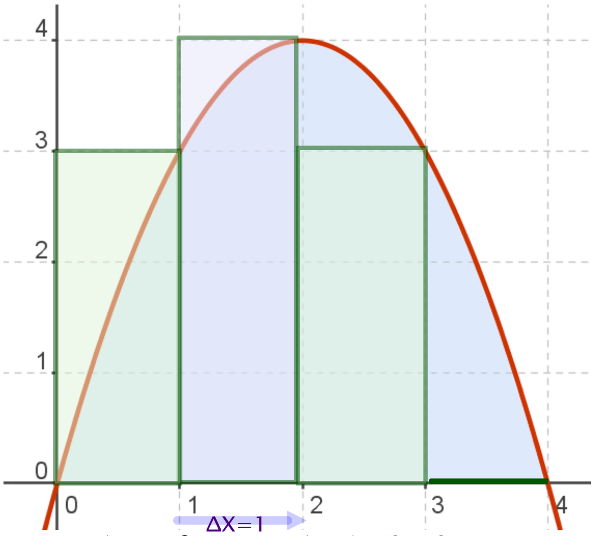
2.1 Area

Using 4 rectangles:



Area of rectange 1 = 1 x 3 = 3
Area of rectange 2 = 1 x 4 =4
Area of rectange 3 = 1 x 3 = 3
Area of rectange 4 = 1 x 0 = 0

$\Sigma = 10$

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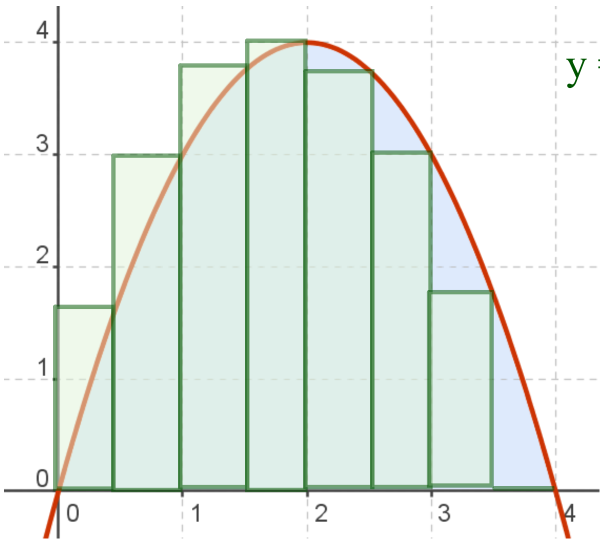
2.1 Area

Using 8 rectangles:



y = -x(x-4)

$$\sum_{i=1}^{8} \Delta x_i f(c_i)$$

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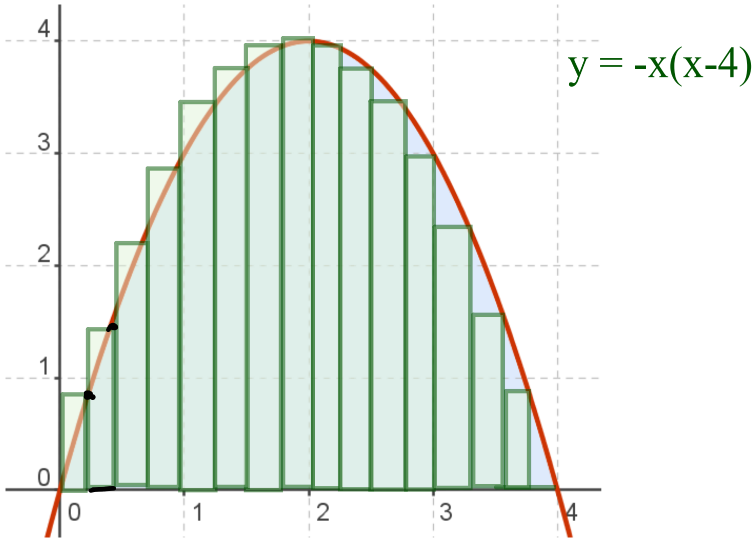
2.1 Area

Using 16 rectangles:



$$\sum_{i=1}^{16} \Delta x_i f(c_i)$$

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2.1 Area

For the following exercises, consider the function $f(x) = \sqrt{1 - x^2}$. (*Hint*: This is the upper half of a circle of radius 1 positioned at $(0, 0)$.)

26. Sketch the graph of f over the interval $[-1, 1]$.

27. Use the preceding exercise to find the exact area between the x-axis and the graph of f over the interval $[-1, 1]$ using rectangles. For the rectangles, use squares 0.4 by 0.4 units, and approximate both above and below the lines. Use geometry to find the exact answer.

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2.1 Area

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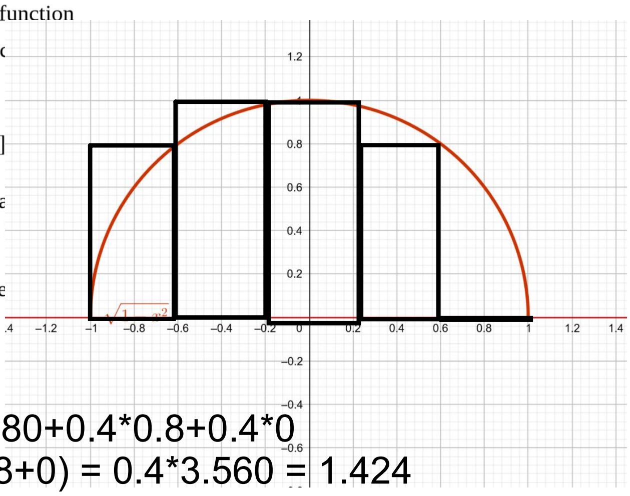

0.4*0.8 + 0.4*0.980+0.4*0.980+0.4*0.8+0.4*0
= 0.4(0.8 + 0.980+0.980+0.8+0) = 0.4*3.560 = 1.424

A = (1/2) π r² = (1/2) π (1) = π/2 = 1.5708 actual

For 10 rectangles of width 0.2 would get:

0.2*f(-0.8)+0.2*f(-0.6)+0.2*f(-0.4)+0.2*f(-0.2)+0.2*f(0)+0.2*f(0.2)+0.2*f(0.4)+0.2*f(0.6)
+0.2*f(0.8)+0.2*f(1) = 0.2[0.6+0.8+0.9165+0.9798+1+0.9798+0.9165+0.8+0.6+0]
=0.2*7.5926= **1.51852** **getting closer to actual**

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