11.5 Alternating Series; 11.6 Convergence, Ratio, Root Tests

Goals:

1. Recognize Alternating Series:
   terms alternate between positive and negative.
2. Apply the Alternating Series Test:
   conditions for convergence of alternating series
3. Conditional and Absolute Convergence:
   compare convergence of alternating series to convergence of series of absolute values
4. Approximate a series' sum, S, by its nth partial sum, s_n.
5. For s_n approx, Remainder: $|R_N| = |S - s_N| < a_{N+1}$
6. Use Ratio Test, esp. for series with factorials.
7. Use Root test, esp. for series with powers of n.

Study 11.5 # 1 - 17, 23

Study 11.6 # 1-13, 19-29, 33, 35, 39

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HW 11.5

11.5 Alternating Series

Alternating Series: Let $a_n > 0$

$$\sum_{n=1}^{\infty} (-1)^n a_n = -a_1 + a_2 - a_3 + a_4 - \ldots$$

$$\sum_{n=1}^{\infty} (-1)^{n+1} a_n = a_1 - a_2 + a_3 - a_4 + \ldots$$

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$\lim_{n\to\infty}$ ∞

HW 11.5

$\sum_{n=1}^{\infty}$

11.5 Alternating Series

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$$\sum_{n=1}^{\infty} (-1)^{n+1} a_n = a_1 - a_2 + a_3 - a_4 + \ldots$$

Alternating Series Test:

$\sum_{n=1}^{\infty} (-1)^n a_n$ $\sum_{n=1}^{\infty} (-1)^{n+1} a_n$ $a_n > 0$ **Converge if:**

1. $\lim_{n\to\infty} a_n = 0$
2. $a_{n+1} \le a_n$ for all n

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11.5 Alternating Series

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$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^3}$$ Converge? or diverge?

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11.5 Alternating Series

Alternating Series Test:

$\sum_{n=1}^{\infty}(-1)^n a_n$ $\sum_{n=1}^{\infty}(-1)^{n+1} a_n$ $a_n > 0$ **Converge if:**

1. $\lim_{n\to\infty} a_n = 0$ 2. $a_{n+1} \leq a_n$ for all n

$$\sum_{n=1}^{\infty}\frac{(-1)^n}{n^3}$$

Converge? or diverge?

$a_n = \frac{1}{n^3} > 0$ $\lim_{n\to\infty}\frac{1}{n^3} = 0$ $\frac{1}{(n+1)^3} \leq \frac{1}{n^3}$

$$\sum_{n=1}^{\infty}\frac{(-1)^n}{n^3}$$

Converges by alternating series test

return

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11.5 Alternating Series

Alternating Series Test:

$\sum_{n=1}^{\infty}(-1)^n a_n$ $\sum_{n=1}^{\infty}(-1)^{n+1} a_n$ $a_n > 0$ **Converge if:**

1. $\lim_{n\to\infty} a_n = 0$ 2. $a_{n+1} \leq a_n$ for all n

$$\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots$$

Similar to harmonic series, but is alternating. Harmonic series diverges. Does this? Converge? or diverge?

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11.5 Alternating Series

Alternating Series Test:

$\sum_{n=1}^{\infty}(-1)^n a_n$ $\sum_{n=1}^{\infty}(-1)^{n+1} a_n$ $a_n > 0$ **Converge if:**

1. $\lim_{n\to\infty} a_n = 0$ 2. $a_{n+1} \le a_n$ for all n

$$\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}$$ Converge? or diverge?

$a_n = \frac{1}{n} > 0$ $\lim_{n\to\infty}\frac{1}{n} = 0$ $\frac{1}{n+1} \le \frac{1}{n}$ for all n

$$\sum_{n=1}^{\infty}\frac{(-1)^{n}}{n}$$ Converges by alternating series test

return

11.5 Alternating Series; 11.6 Convergence, Ratio, Root Tests

Definitions:

Conditionally Convergent: The series Σa_n is conditionally convergent if the series converges, but the series $\Sigma|a_n|$ diverges.

Absolutely Convergent: The series Σa_n is absolutely convergent if the series $\Sigma|a_n|$ converges.

Theorum:
If $\Sigma\,|a_n|$ converges, then $\Sigma\, a_n$ converges.

Remainder: Given a convergent alternating series with sum S, the absolute value of the remainder satisfies:

$$|R_N| = |S - S_N| \le a_{N+1}$$

11.5 Alternating Series; 11.6 Convergence, Ratio, Root Tests

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	Series	$\Sigma\, a_n$	$\Sigma\, \lvert a_n \rvert$	Type, Conv	p
link	$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^3}$				3
link	$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$				1
link	$\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$				½

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DDDD

Conditional Convergence

Conditional Convergence

Absolute Convergence

Absolute Convergence

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11.5 Alternating Series; 11.6 Convergence, Ratio, Root Tests

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Conditionally Convergent: The series Σa_n is conditionally convergent if the series converges, but the series $\Sigma|a_n|$ diverges.

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Series	$\Sigma\, a_n$	$\Sigma\, \lvert a_n \rvert$	Type	p
$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^3}$	C	C	Absolute Convergence	3
$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$	C	D	Conditional Convergence	1
$\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$	C	D	Conditional Convergence	½

∞

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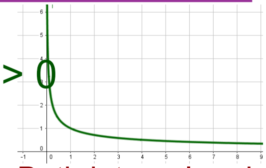
11.3 Integral Test

The Integral Test:
Let **f(x)** > 0, continuous, and decreasing, $x \geq 0$, and $a_n = f(n)$.
Then, $\sum_{n=1}^{\infty} a_n$ and $\int_1^{\infty} f(x)\,dx$
either both converge or both diverge

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} = 1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \ldots$$

Converge? or diverge?

$f(x) = \frac{1}{\sqrt{x}} > 0$, continuous, decreasing for $x > 0$



$$\int_1^{\infty} x^{-1/2}dx = \lim_{b\to\infty}\left[2x^{1/2}\right]_1^b = \lim_{b\to\infty}\left[2\sqrt{b} - 2\right]_1^b$$

∴ Both integral and Series **Diverge**

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\sqrt{n}} \qquad a_n = \frac{1}{\sqrt{n}} > 0 \qquad \lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0 \qquad \frac{1}{\sqrt{n+1}} \leq \frac{1}{\sqrt{n}} \text{ for all n}$$

$$\sum_{n=1}^{\infty} \frac{(-1)^{n}}{\sqrt{n}}$$

Converges by alternating series test

return

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11.5 Alternating Series

Remainder: Given a convergent alternating series with sum S, the absolute value of the remainder satisfies:

$$|R_N| = |S - S_N| \leq a_{N+1}$$

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$$

Converges by alternating series test
What is the sum, S, of the series?
Can we estimate S by using s_n, eg: s_{999}?
What error is associated with the estimate?

sum(seq(Y1,x,1,999)) with Y1 = $(-1)^{x+1}/x$ results in 0.69364743

Can use s_{999} as estimate of S: $S \approx 0.693647$

The error generated by using the estimate is bounded by the value of $s_{1000} = 1/1000$

$$|R_N| = |S - S_N| < a_{N+1} = 1/1000$$

using n=1000 caused an overflow on my calculator

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11.6 Convergence, Ratio, Root Tests

Ratio Test: Let Σa_n be a series with nonzero terms.

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| < 1$$ Then the series converges absolutely

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| > 1$$ Then the series diverges

The test is inconclusive if $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 1$

Especially useful for series having factorials.

$\sum_{n=1}^{\infty}$

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11.6 Convergence, Ratio, Root Tests

$$\sum_{n=1}^{\infty}\frac{2^n}{n!}$$

Converge? or diverge?

all terms nonzero, has factorial, $\therefore$ use ratio test

$\sum_{n=1}^{\infty}$

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11.6 Convergence, Ratio, Root Tests

$$\sum_{n=1}^{\infty} \frac{2^n}{n!}$$

Converge? or diverge?

all terms nonzero, has factorial, ∴ use **ratio test**

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{2^{n+1}/(n+1)!}{2^n/n!}\right| = \lim_{n\to\infty}\left|\frac{2^{n+1}}{2^n}\frac{n!}{(n+1)!}\right|$$

$$= \lim_{n\to\infty}\left|\frac{2}{(n+1)}\right| = 0 < 1$$

$(n+1)! = (n+1)n!$

The series converges absolutely

$\sum_{n=1}^{\infty}$

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∞ HW 11.5 HW 11.6 $\sum_{n=1}^{\infty}$

11.6 Convergence, Ratio, Root Tests

$$\sum_{n=1}^{\infty} n\left(\frac{6}{5}\right)^n$$

Converge? or diverge?

all terms nonzero, not obvious which, try **ratio test**

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|$$

$\sum_{n=1}^{\infty}$

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11.6 Convergence, Ratio, Root Tests

$$\sum_{n=1}^{\infty} n\left(\frac{6}{5}\right)^n$$

Converge? or diverge?

all terms nonzero, not obvious which, try **ratio test**

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)/(6/5)^{n+1}}{n\,/\,(6/5)^n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)}{n}\left(\frac{6}{5}\right)\right|$$

$$= \frac{6}{5} > 1$$

The series diverges

$\sum_{n=1}^{\infty}$

11.6 Convergence, Ratio, Root Tests

Root Test: Let Σa_n be a series

$\lim_{n\to\infty}\sqrt[n]{|a_n|} < 1$ Then the series converges absolutely

$\lim_{n\to\infty}\sqrt[n]{|a_n|} > 1$ Then the series diverges

The test is inconclusive if $\lim_{n\to\infty}\sqrt[n]{|a_n|} = 1$

Especially useful for series involving nth roots.

$\sum_{n=1}^{\infty}$

11.6 Convergence, Ratio, Root Tests

$$\sum_{n=1}^{\infty} \frac{e^{2n}}{n^n}$$

Converge? or diverge?

has powers of n ∴ try **root test**

11.6 Convergence, Ratio, Root Tests

$$\sum_{n=1}^{\infty} \frac{e^{2n}}{n^n}$$

Converge? or diverge?

has powers of n ∴ try **root test**

$$\lim_{n\to\infty} \sqrt[n]{\frac{e^{2n}}{n^n}} = \lim_{n\to\infty} \left(\frac{e^{2n}}{n^n}\right)^{1/n} = \lim_{n\to\infty} \frac{e^2}{n} = 0 < 1$$

The series converges absolutely

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11.6 Convergence, Ratio, Root Tests

$$\sum_{n=1}^{\infty}\left(\frac{2n+1}{n-1}\right)^n$$

Converge? or diverge?

has powers of n ∴ try **root test**

11.6 Convergence, Ratio, Root Tests

$$\sum_{n=1}^{\infty}\left(\frac{2n+1}{n-1}\right)^n$$

Converge? or diverge?

has powers of n ∴ try **root test**

$$\lim_{n\to\infty}\sqrt[n]{\left(\frac{2n+1}{n-1}\right)^n} = \lim_{n\to\infty}\frac{2n+1}{n-1} = 2 > 1$$

The series diverges

11.6 Convergence, Ratio, Root Tests

Note:

Neither the Ratio Test nor The Root Test work for p-series.

$\sum_{n=1}^{\infty}$ $\sqrt{}$