11.10 Taylor and Maclaurin Series

Goals:
1. Recognize *Taylor Series*.
2. Recognize the Maclaurin Series.
3. Derive Taylor series and Maclaurin series representations for known functions.

Study 11.10 # 1-11, 15, 19

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11.10 Taylor and Maclaurin Series

Taylor Series for f at c:

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(x-c)^n = f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \ldots + \frac{f^{(n)}(c)}{n!}(x-c)^n + \ldots$$

centered at c

Maclaurin Series for f at c=0:

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \ldots + \frac{f^{(n)}(0)}{n!}x^n + \ldots$$

centered at 0

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11.10 Taylor and Maclaurin Series

Power Series: Infinite polynomial.

$\sum_{n=0}^{\infty} b_n(x-c)^n = b_0 + b_1(x-c) + b_2(x-c)^2 + \ ...$ centered at c

Explore derivatives:

$f(x) = b_0 + b_1(x-c) + b_2(x-c)^2 + b_3(x-c)^3 \ ...$

$f\,'(x) = 0 + b_1 + 2b_2(x-c) + 3b_3(x-c)^2 + ... + nb_n(x-c)^{n-1} + ..$

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11.10 Taylor and Maclaurin Series

Power Series: Infinite polynomial.

$\sum_{n=0}^{\infty} b_n(x-c)^n = b_0 + b_1(x-c) + b_2(x-c)^2 + \ ...$ centered at c

Explore derivatives:

$f(x) = b_0 + b_1(x-c) + b_2(x-c)^2 + b_3(x-c)^3 \ ...$

$f\,'(x) = 0 + b_1 + 2b_2(x-c) + 3b_3(x-c)^2 + ... + nb_n(x-c)^{n-1} + ..$

$f\,''(x) = 0 + 0 + 2b_2 + 3 \cdot 2b_3(x-c) + ... + n(n-1)b_n(x-c)^{n-2} + ..$

$f\,'''(x) = 0 + 0 + 0 + 3 \cdot 2b_3 + 4 \cdot 3 \cdot 2b_4(x-c) + ...$
$\qquad + n(n-1)(n-2)b_n(x-c)^{n-3} + ..$

$f^n(x) = n(n-1)(n-2) \cdot\cdot 3 \cdot 2 \cdot 1 \ b_n(x-c)^{n-n} + ..$

$f^n(x) = n!b_n(x-c)^0 + ..$

Let x=c: terms in +... are 0 (have x-c)
Solve for b_n:

$f^n(x) = n!b_n + ..$ → $b_n = \dfrac{f^n(c)}{n!}$

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11.10 Taylor and Maclaurin Series

Power Series: Infinite polynomial.

$$\sum_{n=0}^{\infty} b_n(x-c)^n = b_0 + b_1(x-c) + b_2(x-c)^2 + \ \ldots$$

centered at c

Substitute b_n : $b_n = \frac{f^n(c)}{n!}$

$$\sum_{n=0}^{\infty} \frac{f^n(c)}{n!}(x-c)^n = f(c) + f'(c)(x-c) + \frac{f''(c)}{2}(x-c)^2 + \frac{f'''(c)}{3\cdot 2}(x-c)^3 + \ldots$$

Taylor Series for f at c:

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11.10 Taylor and Maclaurin Series

Taylor Series for f at c:

$$\sum_{n=0}^{\infty} \frac{f^n(c)}{n!}(x-c)^n = f(c) + f'(c)(x-c) + \frac{f''(c)}{2}(x-c)^2 + \frac{f'''(c)}{3\cdot 2}(x-c)^3 + \ldots$$

centered at c

Can we use this to represent ***f(x) = ln x*** as a series *centered at 1*?

need f '(x), f "(x), etc. and f '(1), f "(1), etc

$f(x) = \ln x$	$f(1) = \ln 1 = 0$
$f'(x) =$	$f'(1) =$
$f''(x) =$	$f''(1) =$
$f'''(x) =$	$f'''(1) =$
$f^{(4)}(x) =$	$f^{(4)}(1) =$
	$f^n(1) =$

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$\sum_{n=0}^{\infty} \sum_{=0}^{\infty}$

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11.10 Taylor and Maclaurin Series

Taylor Series for f at c:

$$\sum_{n=0}^{\infty} \frac{f^{n}(c)}{n!}(x-c)^n = f(c) + f'(c)(x-c) + \frac{f''(c)}{2}(x-c)^2 + \frac{f'''(c)}{3\cdot 2}(x-c)^3 + \dots$$

centered at c

Can we use this to represent ***f(x) = ln x*** as a series *centered at 1*?

need f '(x), f "(x), etc. and f '(1), f "(1), etc

$f(x) = \ln x$	$f(1) = \ln 1 = 0$
$f'(x) = 1/x$	$f'(1) = 1/1 = 1$
$f''(x) = -1/x^2$	$f''(1) = -1/1^2 = -1$
$f'''(x) = 2/x^3$	$f'''(1) = 2/1^3 = 2$
	$f^{n}(1) = (-1)^{n+1}(n-1)!$

$$f(x) = \ln x = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}(n-1)!}{n!}(x-1)^n = 0 + (x-1) - \frac{(x-1)^2}{2} + \frac{2(x-1)^3}{3\cdot 2} + \dots$$

$$f(x) = \ln x = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}(x-1)^n = (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \dots$$

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11.10 Taylor and Maclaurin Series

Taylor Series for f at c:

$$\sum_{n=0}^{\infty} \frac{f^{n}(c)}{n!}(x-c)^n = f(c) + f'(c)(x-c) + \frac{f''(c)}{2}(x-c)^2 + \frac{f'''(c)}{3\cdot 2}(x-c)^3 + \dots$$

centered at c

$$f(x) = \ln x = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}(x-1)^n = (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \dots$$

Compare to result in 11.9, power series: centered at 1

$$\ln x = \sum_{n=0}^{\infty} (-1)^n \frac{(x-1)^{n+1}}{n+1} = \frac{x-1}{1} - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4} + \dots$$

Can we represent ***ln x*** as a **power series?**
Consider **Integration**:

$$\frac{1}{x} = \sum_{n=0}^{\infty} (-1)^n (x-1)^n$$

$$\int \frac{1}{x}\,dx = \int \sum_{n=0}^{\infty} (-1)^n (x-1)^n \,dx$$

$$\ln x + c = \sum_{n=0}^{\infty} (-1)^n \frac{(x-1)^{n+1}}{n+1}$$

let x=1, $\ln 1 + c = \sum_{n=0}^{\infty} 0 = 0$

$$\ln x = \sum_{n=0}^{\infty} (-1)^n \frac{(x-1)^{n+1}}{n+1}$$

Thus, ***ln x*** can be represented as a power series.

Find: R and IC, Use Ratio Test:

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(x-1)^{n+2}/(n+2)}{(x-1)^{n+1}/(n+1)}\right| = \lim_{n\to\infty}\left|\frac{(x-1)(n+1)}{(n+2)}\right| = |x-1|$$

R=1, as before integration, but need to check endpoints $|x-1| < 1$ $0 < x < 2$

let x=0: Limit Comparison Test

$$\lim_{n\to\infty}\frac{1/(n+1)}{1/n} = \lim_{n\to\infty}\frac{n}{(n+1)} = 1$$

since 1/n diverges, so does 1/(n+1) by Limit Comparison Test

let x=2 $\lim_{n\to\infty}\frac{(-1)^n}{n+1} = 0$

alternating series
$1/(n+2) < 1/(n+1)$
converges at x = 2

IC: $0 < x \le 2$
gained a pt.

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11.10 Taylor and Maclaurin Series

Taylor Series for f at c:

$$\sum_{n=0}^{\infty} \frac{f^n(c)}{n!}(x-c)^n = f(c) + f'(c)(x-c) + \frac{f''(c)}{2}(x-c)^2 + \frac{f'''(c)}{3 \cdot 2}(x-c)^3 + \ldots$$

centered at c

$$\sum_{n=1}^{\infty} \qquad f(x) = \ln x = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}(x-1)^n = (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \ldots$$

Compare to result in 11.9, power series: centered at 1

$$\ln x = \sum_{n=0}^{\infty} (-1)^n \frac{(x-1)^{n+1}}{n+1} = \frac{x-1}{1} - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4} + \ldots$$

$0 < x \le 2$

1. Get same result.
2. Using the Taylor Series avoids the need to recognize which function to start manipulating (integrating).
3. Either way, still need to identify R and IC, the Radius and Interval of Convergence.

$$\frac{f^n(x)}{n!} \qquad \sum_{n=0}^{\infty}$$

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11.10 Taylor and Maclaurin Series

Taylor Series for f at c:

$$\sum_{n=0}^{\infty} \frac{f^n(c)}{n!}(x-c)^n = f(c) + f'(c)(x-c) + \frac{f''(c)}{2}(x-c)^2 + \frac{f'''(c)}{3 \cdot 2}(x-c)^3 + \ldots$$

centered at c

Represent ***f(x) = e^x*** as a series *centered at 0*

$f(x) = e^x$	$f(0) = e^0 = 1$
$f'(x) =$	$f'(0) =$
$f''(x) =$	$f''(0) =$
$f'''(x) =$	$f'''(0) =$
$f^n(x) =$	$f^n(0) =$

centered at 0

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11.10 Taylor and Maclaurin Series

Taylor Series for f at c:

$$\sum_{n=0}^{\infty} \frac{f^n(c)}{n!}(x-c)^n = f(c) + f'(c)(x-c) + \frac{f''(c)}{2}(x-c)^2 + \frac{f'''(c)}{3\cdot 2}(x-c)^3 + \dots$$

centered at c

Represent $f(x) = e^x$ as a series *centered at 0*

$f(x) = e^x$	$f(0) = e^0 = 1$
$f'(x) = e^x$	$f'(0) = e^0 = 1$
$f''(x) = e^x$	$f''(0) = e^0 = 1$
$f'''(x) = e^x$	$f'''(0) = e^0 = 1$
$f^n(x) = e^x$	$f^n(0) = e^0 = 1$

centered at 0

$$f(x) = e^x = \sum_{n=0}^{\infty} \frac{1}{n!} x^n = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \dots + \frac{x^n}{n!} + \dots$$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{x^{n+1}/(n+1)!}{x^n/n!}\right| = \lim_{n\to\infty}\left|\frac{x^{n+1}\, n!}{x^n\,(n+1)!}\right| = \lim_{n\to\infty}\left|\frac{x}{n+1}\right| = 0 < 1$$

$f(x) = e^x$ converges for all x: *$R = \infty$, IC: $(-\infty, \infty)$*

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11.10 Taylor and Maclaurin Series

Taylor Series for f at c:

$$\sum_{n=0}^{\infty} \frac{f^n(c)}{n!}(x-c)^n = f(c) + f'(c)(x-c) + \frac{f''(c)}{2}(x-c)^2 + \frac{f'''(c)}{3\cdot 2}(x-c)^3 + \dots$$

centered at c

$$f(x) = e^x = \sum_{n=0}^{\infty} \frac{1}{n!} x^n = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \dots + \frac{x^n}{n!} + \dots$$

centered at 0

Used c =0, so actually a Maclaurin Series

Maclaurin Series for f at c=0:

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \dots + \frac{f^{(n)}(0)}{n!}x^n + \dots$$

centered at 0

11.10 Taylor and Maclaurin Series

Maclaurin Series for f at c=0:

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = f(0)+f'(0)x + \frac{f''(0)}{2!}x^2 + \dots + \frac{f^{(n)}(0)}{n!}x^n+\dots$$ centered at 0

Find the Maclaurin Series for **f(x) = sin x**.

$f(x) = \sin x$	$f(0) = \sin 0 = 0$
$f'(x) =$	$f'(0) =$
$f''(x) =$	$f''(0) =$
$f'''(x) =$	$f'''(0) =$
$f^{(4)}(x) =$	$f^{(4)}(0) =$

$f^n(0) = ?$ *write out some terms and look for patterns*

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Maclaurin Series for f at c=0:

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = f(0)+f'(0)x + \frac{f''(0)}{2!}x^2 + \dots + \frac{f^{(n)}(0)}{n!}x^n+\dots$$ centered at 0

Find the Maclaurin Series for **f(x) = sin x**.

$f(x) = \sin x$	$f(0) = \sin 0 = 0$
$f'(x) = \cos x$	$f'(0) = \cos 0 = 1$
$f''(x) = -\sin x$	$f''(0) = -\sin 0 = 0$
$f'''(x) = -\cos x$	$f'''(0) = -\cos 0 = -1$
$f''''(x) = \sin x$	$f''''(0) = \sin 0 = 0$

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = 0 + x + \frac{0}{2!}x^2 - \frac{x^3}{3!} + \frac{0x^4}{4!} + \frac{x^5}{5!} \quad \dots + \frac{f^{(n)}(0)}{n!}x^n+\dots$$

$$\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots + \frac{(-1)^n x^{2n+1}}{(2n+1)!} + \dots$$

n 0 1 2 n

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11.10 Taylor and Maclaurin Series

Find R, IC for f(x) = sin x.

$$\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots + \frac{(-1)^n x^{2n+1}}{(2n+1)!} + \dots$$

centered at 0

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\underline{\qquad\qquad}\right| = \lim_{n\to\infty}\left|\underline{\qquad\qquad}\right| = \lim_{n\to\infty}\left|\underline{\qquad\qquad}\right|$$

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11.10 Taylor and Maclaurin Series

Find R, IC for f(x) = sin x.

$$\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots + \frac{(-1)^n x^{2n+1}}{(2n+1)!} + \dots$$

centered at 0

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{x^{2n+3}/(2n+3)!}{x^{2n+1}/(2n+1)!}\right| = \lim_{n\to\infty}\left|\frac{x^{2n+3}\,(2n+1)!}{x^{2n+1}(2n+3)!}\right| = \lim_{n\to\infty}\left|\frac{x^2}{(2n+3)(2n+2)}\right|$$

$$= 0 < 1$$

f(x) = sin x converges for all x: R = ∞, IC: (-∞, ∞)

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$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = f(0)+f'(0)x + \frac{f''(0)}{2!}x^2 + \ldots + \frac{f^{(n)}(0)}{n!}x^n+\ldots$$ centered at 0

Find the Maclaurin Series for **f(x) = cos x**.

$$f(x)=\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots + \frac{(-1)^n x^{2n+1}}{(2n+1)!} + \ldots$$ centered at 0

$$f'(x)=\cos x = \frac{d}{dx}\left[\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}\right] =$$

$$f'(x)=\cos x =$$

=

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11.10 Taylor and Maclaurin Series

Find the Maclaurin Series for **f(x) = cos x**.

$$f(x)=\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots + \frac{(-1)^n x^{2n+1}}{(2n+1)!} + \ldots$$ centered at 0

$$f'(x)=\cos x = \frac{d}{dx}\left[\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}\right] = \sum_{n=0}^{\infty} \frac{(-1)^n(2n+1)\, x^{2n}}{(2n+1)!}$$

$$f'(x)=\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} = 1 - \frac{x^2}{2} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \ldots$$

=

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11.10 Taylor and Maclaurin Series

Find R, IC for f(x) = cos x.

$$f'(x) = \cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} = 1 - \frac{x^2}{2} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \ldots$$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\right| = \lim_{n\to\infty}\left|\right| = \lim_{n\to\infty}\left|\right|$$

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11.10 Taylor and Maclaurin Series

Find R, IC for f(x) = cos x.

$$f'(x) = \cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} = 1 - \frac{x^2}{2} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \ldots$$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{x^{2n+1}/(2n+2)!}{x^{2n}/(2n)!}\right| = \lim_{n\to\infty}\left|\frac{x^{2n+1}(2n)!}{x^{2n}(2n+2)!}\right| = \lim_{n\to\infty}\left|\frac{x}{(2n+2)(2n+1)}\right|$$

$$= 0 < 1$$

f(x) = cos x converges for all x: *R = ∞, IC: (-∞, ∞)*

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11.10 Taylor and Maclaurin Series

Find the Maclaurin Series for **$f(x) = (1 + x)^k$**, k real number

$f(x) = (1+x)^k$	$f(0) = 1$
$f'(x) = k(1+x)^{k-1}$	$f'(0) = k$
$f''(x) = k(k-1)(1+x)^{k-2}$	$f''(0) = k(k-1)$
$f'''(x) = k(k-1)(k-2)(1+x)^{k-3}$	$f'''(0) = k(k-1)(k-2)$
$f''''(x) = k(k-1)(k-2)(k-3)(1+x)^{k-4}$	$f''''(0) = k(k-1)(k-2)(k-3)$

$f^{(n)}(x) = k(k-1)(k-2)(k-3)\cdots(k-n+1)(1+x)^{k-n}$

$f^{(n)}(0) = k(k-1)(k-2)(k-3)\cdots(k-n+1)$

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = \sum_{n=0}^{\infty} \frac{k(k-1)(k-2)\cdots(k-n+1)}{n!} x^n = \sum_{n=0}^{\infty} \binom{k}{n} x^n$$

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11.10 Taylor and Maclaurin Series

Find R, IC for $f(x) = (1 + x)^k$

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = \sum_{n=0}^{\infty} \frac{k(k-1)(k-2)\cdots(k-n+1)}{n!} x^n = \sum_{n=0}^{\infty} \binom{k}{n} x^n$$

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{k(k-1)\ldots(k-n)x^{n+1}/(n+1)!}{k(k-1)\ldots(k-n+1)\,x^n/n!}\right| = \lim_{n\to\infty} \left|\frac{(k-n)\,x}{n+1}\right|$$

$$= |x| < 1$$

$f(x) = (1 + x)^k$ *converges* $|x|<1$ $R = 1$, *IC:* $(-1, 1)$

Includes endpoints when:

$-1 < k < 0$ converges **(-1, 1]**

$k \geq 0$ converges **[-1, 1]**

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